ABSTRACT

A study is made of the numbers G_n defined by $\sum_{n=1}^{\infty} \frac{G_n x^n}{n!} = e^{e^x - 1}$. The results of numerous papers dealing with these numbers are correlated. The equivalence of several definitions of the numbers is proved and many of their arithmetic and analytic properties are derived. Tables for the numbers G_n and certain related arithmetic functions are given. Some of these considerably extend the range of previous tabulations.

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NUMBERS GENERATED BY THE FUNCTION e^{e^x-1}.

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INTRODUCTION

In this thesis we consider a set of numbers G_n, defined by $e^{e^x-1} = \sum_{n=1}^{\infty} \frac{G_n x^n}{n!}$. Although these G's have been considered by numerous mathematicians, the results obtained have never been correlated. Such a correlation is the main object of this dissertation.

In Chapter 1 we discuss several problems, all of which have the numbers G_n as their solution. This leads to a number of defining properties of the G's. A relation between the G's and Stirling numbers of the second kind is established. From these we obtain several representations of the G's as finite sums.

In Chapter 2 we derive recursion formulas for the G's. We define a double sequence of numbers A_{mn} and show how this may be used to compute the G's. We obtain arithmetic properties of the A's and G's and several determinantal forms of the latter.

In Chapter 3 we generalize the function G_n to a function of a complex variable. Various expansions for this function are obtained and special cases of these are considered. We conclude the chapter with a derivation of an asymptotic formula for G_n.

In Chapter 4 we summarize the results of many isolated notes and papers dealing with the G's and closely related numbers and functions.

In Chapter 5 we consider the manner in which the G's appear in two applied problems. The first of these deals with the number of measurable impedances for an n-terminal network while the second is concerned with a particular probability distribution.

The thesis concludes with a presentation of tables of the G's and related arithmetic functions. Some of these represent a considerable extension over previously published results.

CHAPTER 1

SOME DEFINING PROPERTIES OF THE G'S.

In this chapter we state four problems which have a common solution. This solution is G_n , the coefficient of $\frac{x^n}{n!}$ in the expansion of $e^{e^x - 1}$.

First, we state these four problems. We then show that the solutions for these problems are equivalent and we give four methods of solution. Each solution has a different form; the final section of the chapter proves directly that two of the forms are equal.

The problems referred to above are:

1.1 How many ways are there of putting n distinguishable objects into 1, 2, ..., n indistinguishable parcels? (By a parcel is meant a set in which the arrangement of the articles in the set is not considered).

1.2 What is the number of abstract non-isomorphic equivalence relations among n elements? (An equivalence relation is a relation which is reflexive, transitive and symmetric. Such a relation has the effect of separating a set of elements intodisjoint classes. By an abstract equivalence relation we mean the specification by which the elements are put in the various classes).

1.3 In how many ways can the product of n distinct primes be factored?

1.4 How many rhyming schemes are there for a stanza of n lines? Sylvester [45] considered this problem.

Theorem 1.1: The four problems 1.1 - 1.4 are equivalent.

Proof: An abstract equivalence relation is defined among a set of elements when the elements are divided into different sets and then the individuals in each set are treated as equivalent. Hence, the number of such relations is the same as the number of ways of putting n distinguishable objects into 1, 2, ..., n indistinguishable parcels. Similarly, the number of ways of factoring the product of n distinct primes is the number of ways of breaking up the n primes into smaller sets and so is, again, the same as the number of ways of putting n distinguishable objects into 1, 2, ..., n indistinguishable parcels. In exactly the same way it is seen that the number of rhyming schemes for a stanza of n lines is the number of ways of separating last words of lines into various sets such that all the words, in one set, rhyme. Hence, this problem is the same as the first one mentioned.

We will consider problem 1.1 and obtain the solution G_n in a particular form.

Several papers have been written on the use of finite difference operators in the symbolic solution of card matching problems [30] [31] [37]. Mendelsohn [38], by using finite difference operators, obtains two explicit formulae for the G's. To obtain these formulae we use the standard operators E and Δ defined by

$$E u_n = u_{n+1} \quad ; \quad \Delta u_n = u_{n+1} - u_n .$$

Clearly

$$E = 1 + \Delta .$$

Suppose we are given the set of equations

1.5 $$v_i = \sum_{j=0}^{i} c_{ij} u_j \quad (i = 0, 1, 2, \ldots, n)$$

subject to the condition

$$c_{00} c_{11} c_{22} \cdots c_{nn} \neq 0 .$$

Then

$$v_0 = c_{00} u_0$$

so that

$$u_0 = \frac{1}{c_{00}} v_0 .$$

Now, since we know u_0 we can solve for u_1 in the equation 1.5

Similarly we solve successively for $u_1, \cdots u_n$. Thus, the set of equations 1.5 can be solved for u_i.

$$u_i = \sum_{j=0}^{i} d_{ij} v_j .$$

To relate the d_{ij} and c_{ij} symbolically we write

1.6 $$v_n = P_n(E) u_0 \qquad (n = 0, 1, 2, \ldots m).$$

and

1.7 $$u_n = Q_n(E) v_0 \qquad (n = 0, 1, 2, \ldots m).$$

where

$$P_n(E) = \sum_{i=0}^{n} c_{ni} E^i$$

and

$$Q_n(E) = \sum_{i=0}^{n} d_{ni} E^i .$$

Lemma: 1.8

1.8 $$P_n(Q) = E^n = Q_n(P).$$

Proof: Equation 1.7 implies

$$E^n u_0 = E^n Q_n(E) v_0 = d_{n0} v_0 + d_{n1} v_1 + \cdots + d_{nn} v_n .$$

Hence, using 1.6 the above implies

$$E^n u_0 = [d_{n0} P_0(E) + \cdots + d_{nn} P_n(E)] u_0 .$$

or symbolically

$$E^n = Q_n(P).$$

where in the right hand side we replace P^i by $P_i(E)$.

Similarly

$$E^n = P_n(Q).$$

Theorem 1.2

$$G_n = \sum_{i=1}^{n} \sum_{j=0}^{i} (-1)^j \frac{(i-j)^n}{j!\,(i-j)!} \quad .$$

Proof: One pair of sets of polynomials satisfying 1.8 is

1.9 $P_n(E) = (1+E)^n \quad ; \quad Q_n(E) = \Delta^n$.

Now we consider the number of ways in which n distinguishable objects can be placed into m distinguishable boxes so that each box contains one object at least, and the order of arrangement in the boxes is irrelevant. Let u_{mn} be the required number. The number of ways of placing the objects into the boxes without restriction is m^n . This can be decomposed into the following exclusive cases: no box empty, exactly one box empty, exactly two boxes empty, etc. Then it follows that

$$m^n = u_{mn} + \binom{m}{1} u_{m-1,n} + \binom{m}{2} u_{m-2,n} + \cdots + \binom{m}{m} u_{0,n}.$$

or

$$m^n = (1+E)^m u_{0n}.$$

Thus, by 1.9 we have

$$u_{mn} = \Delta^m 0^n.$$

But

$$\Delta = E - 1$$

so

$$u_{mn} = (E-1)^m 0^n$$

Hence

$$u_{mn} = m^n - \binom{m}{1}(m-1)^n + \ldots (-1)^{m-1}\binom{m}{m-1}.$$

Since u_{mn} is the number of ways of placing n distinguishable objects into m distinguishable boxes, $\frac{u_{mn}}{m!}$ is the number of ways of placing n distinguishable objects into m indistinguishable boxes. Thus, if we let G_n be the number of ways of placing n distinguishable objects into any number of indistinguishable boxes, then

1.10 $$G_n = \sum_{i=1}^{n} \frac{u_{in}}{i!} = \sum_{i=1}^{n} \frac{\Delta^i 0^n}{i!}$$

or

$$G_n = \sum_{i=1}^{n} \sum_{j=0}^{i} (-1)^j \frac{(i-j)^n}{j!\,(i-j)!} .$$

Before considering Jordan's solution [28] for the problem of how many factorizations of the product of n primes there are, we give a definition and theorem

Definition 1.1 :

$${}_mS_n = \frac{\Delta^m 0^n}{m!} .$$

Theorem 1.3 :

$${}_mS_{n+1} = {}_{m-1}S_n + m\,{}_mS_n .$$

Proof: By Jordan

$$\Delta^m(uv) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i v(x+m-i)\, \Delta^{m-i} u.$$

We set $v = x$ and $u = x^n$.

$$\Delta^m (x^{n+1}) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i (x+m-i) \Delta^{m-i} x^n$$

$$= m \Delta^m x^n + m \Delta^{m-1} x^n.$$

Dividing both sides by $m!$ and setting $x = 0$ we have

$$\frac{\Delta^m 0^{n+1}}{m!} = m \frac{\Delta^m 0^n}{m!} + \frac{m \Delta^{m-1} 0^n}{m!}$$

or

$$_m S_{n+1} = m({}_m S_n) + {}_{m-1} S_n .$$

Definition 1.1 clearly implies

$$_0 S_0 = 1, \quad {}_m S_0 = 0 \quad \text{if} \quad m \neq 0.$$

The numbers $_m S_n$ are of considerable importance in the calculus of finite differences [47] and are known as Stirling numbers of the second kind.

Theorem 1.4:

$$G_n = \sum_{m=1}^{n} {}_m S_n .$$

Proof: This follows immediately from 1.10 and Definition 1.1.

Theorem 1.5: The total number of factorizations of the product of n distinct primes is

$$\sum_{m=1}^{n} {}_m S_n = G_n .$$

Proof: Suppose we are given a number ω_n the product of n distinct primes so that

$$\omega_n = \alpha_1 \alpha_2 \cdots \alpha_n .$$

We will let $f(m,n)$ be the number of ways in which ω_n can be decomposed into m factors, not considering permutations of the same factors as being different decompositions. e.g. if

$$\omega_3 = \alpha_1 \alpha_2 \alpha_3$$

$$f(1,3) = 1 \qquad (\alpha_1 \alpha_2 \alpha_3)$$
$$f(2,3) = 3 \qquad (\alpha_1 \alpha_2)(\alpha_3), (\alpha_1)(\alpha_2 \alpha_3), (\alpha_1 \alpha_3)(\alpha_2)$$
$$f(3,3) = 1 \qquad (\alpha_1)(\alpha_2)(\alpha_3) .$$

We can use the decompositions of ω_3 to find the decompositions of ω_4 as follows. If α_4 is adjoined as a separate factor to a decomposition of ω_3 into two factors, a decomposition of ω_4 into three factors will result. If α_4 is adjoined to any of the factors of a decomposition of ω_3 into three factors a decomposition of ω_4 into three factors will result. Thus, we have

$$f(3,4) = f(2,3) + 3f(3,3) .$$

In general, we see that we obtain $f(m,n)$ from $f(m-1,n-1)$ and $f(m,n-1)$ by first adjoining the factor α_n to the decompositions $f(m-1,n-1)$, and secondly, by multiplying successively every factor of the decomposition $f(m,n-1)$

by α_n ; therefore, each of these decompositions will give m new ones, so that we shall have

1.11 $$f(m,n) = f(m-1, n-1) + m f(m, n-1).$$

But, by definition 1.1 this is the difference equation which the Stirling numbers of the second kind satisfy. The initial conditions are the same since

$$f(0,0) = 1 \quad , \quad f(m,0) = 0 \quad \text{if } m \neq 0.$$

We now consider Aitken's [1] method of finding the number of ways of placing n individuals into various sets.

Theorem 1.6: C_n , i.e., the number of ways of arranging n individuals in various sets, is the coefficient of $\frac{x^n}{n!}$ in the expansion of

$$e^{e^x - 1}.$$

Proof: First we note that three individuals A, B, C can be arranged in the following ways:

$$(A)(B)(C)\ ;\ (A)(BC),\ (B)(CA),\ (C)(AB)\ ;\ (ABC).$$

Let a class of n individuals be decomposed into sets as follows: α sets containing a individuals, β sets containing b individuals, etc. We shall denote such a decomposition by

1.12 $$a^\alpha b^\beta c^\gamma \cdots \quad .$$

Clearly we have

$$\alpha a + \beta b + \gamma c + \cdots = n.$$

The number of possible decompositions of the type 1.12 is given by the expression

$$\frac{n!}{(a!)^{\alpha}(b!)^{\beta}(c!)^{\gamma}\cdots\ \alpha!\,\beta!\,\gamma!\cdots}\ .$$

Thus the total number of decompositions of n individuals into sets is

$$1.13 \qquad \sum \frac{n!}{(a!)^{\alpha}(b!)^{\beta}(c!)^{\gamma}\cdots\alpha!\,\beta!\,\gamma!\cdots}\ .$$

Where the summation runs over all possible decompositions of the type 1.12. As an example we give all the decompositions of 4 individuals A, B, C, D into the decompositions $1^4;\ 2^2;\ 3^1 1^1;\ 2^1 1^2;\ 4^1,$ respectively.

1^4: (A)(B)(C)(D)

2^2: (A B)(C D)
(A C)(B D)
(A D)(B C)

$3^1 1^1$: (A B C)(D)
(A B D)(C)
(A C B)(D)
(B C D)(A)

$2^1 1^2$: (A B)(C)(D)
(A C)(B)(D)
(A D)(B)(C)
(B C)(A)(D)
(B D)(A)(C)
(C D)(A)(B)

4^1: (A B C D)

Next, we will find a generating function which has the total number of decompositions of n individuals into sets as the coefficient of $\frac{x^n}{n!}$ in its power series expansion.

This number will be a function of n, which, as we have seen, is G_n .

Any decomposition in which there are r unit sets will contribute to the sum 1.13 a term in which $(1!)^r r!$ appears in the denominator. Hence, let us form the series

$$1.14 \quad \sum_{r=0}^{\infty} \frac{(x^1)^r}{(1!)^r r!} = 1 + \frac{(x^1)^1}{1!} + \frac{(x^1)^2}{2!} + \cdots \quad .$$

We will associate the exponent of x^1 in any one of these terms in the above series with that part of n which is made up by a class of unit sets, the number of unit sets in the class being equal to the exponent considered. Similarly, any decomposition in which there are s sets of size two will contribute to the sum 1.13 a term in which $(2!)^s s!$ appears in the denominator. Thus, we form the series

$$1.15 \quad \sum_{s=0}^{\infty} \frac{(x^2)^s}{(2!)^s s!} \quad .$$

Here we associate the exponent of x^2 in any one of the terms in the series with that part of n which is made up of sets of size two, the number of such sets being again equal to the exponent considered. Similar series are constructed for sets of all sizes up to and including n. Now, if we multiply all these series together, the coefficient of x^n will be the sum of all possible fractions with 1 for numerator and denominators

of the type shown in the series 1.13 and subject to the condition that

$$\alpha a + \beta b + \gamma c + \cdots = n .$$

Thus, the total number of decompositions of n individuals into sets is the coefficient of x^n in the power series formed by multiplying together the series 1.14, 1.15 etc. But we note that

$$\sum_{\alpha=0}^{\infty} \frac{(x^a)^\alpha}{(a!)^\alpha(\alpha!)} = e^{\frac{x^a}{a!}}$$

so that

$$1.16 \quad \sum_{r=0}^{\infty} \frac{(x^1)^r}{(1!)^r r!} \cdot \sum_{s=0}^{\infty} \frac{(x^2)^s}{(2!)^s s!} \cdot \sum_{t=0}^{\infty} \frac{(x^3)^t}{(3!)^t t!} \cdots$$

$$= e^{\frac{x^1}{1!} + \frac{x^2}{2!} + \cdots} .$$

Furthermore, we note that there is no reason why we should not carry the product indicated in 1.16 beyond the n^{th} term since only the term in x^0 in these succeeding terms will contribute anything to x^n in the product. The number of decompositions of n individuals into sets is therefore the coefficient of $\frac{x^n}{n!}$ (in the power series expansion of e^{e^x-1} (c.f. 1.13 for the $n!$ appearing in the denominator here)

or the power series expansion of

$$e^{e^x-1} \quad .$$

Theorem 1.7:

$$G_n = \frac{1}{e}\sum_{t=0}^{\infty} \frac{t^n}{t!} \quad .$$

Proof: By Theorem 1.6 we see that G_n is the coefficient of $\frac{x^n}{n!}$ in the Maclaurin expansion of e^{e^x-1}. This expansion is 1.17

$$e^{e^x-1} = \frac{1}{e}\, e^{e^x} = \frac{1}{e}\sum_{r=0}^{\infty} \frac{e^{xr}}{r!}$$

$$= \frac{1}{e}\sum_{r=0}^{\infty} \frac{1}{r!}\sum_{n=0}^{\infty} \frac{x^n r^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\frac{1}{e}\sum_{r=0}^{\infty} \frac{r^n}{r!}\right) \quad .$$

Theorem 1.8:

$$\frac{1}{e}\sum_{r=0}^{\infty} \frac{r^n}{r!} = \sum_{r=0}^{n} \frac{\Delta^r 0^n}{r!} \quad .$$

Proof: This follows directly from 1.10 and Theorem 1.7 but we shall also prove it directly. If we expand t^n in Theorem 1.8 in a series of "factorials"

$$(t)_r = (t)(t-1)(t-2)\cdots(t-r+1)$$

we have, according to Jordan [21]

$$t^n = \sum_{r=0}^{n} \frac{\Delta^r 0^n}{r!}\,(t)_r \quad .$$

This then gives with 1.17

$$e^{e^x-1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\frac{1}{e} \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{s=0}^{n} \frac{\Delta^s 0^n}{s!} (t)_s \right).$$

Hence, the coefficients in the Maclaurin expansion of e^{e^x-1} are

$$\frac{1}{e} \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{s=0}^{n} \frac{\Delta^s 0^n}{s!} (t)_s = \frac{1}{e} \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{s=0}^{n} \binom{t}{s} \frac{\Delta^s 0^n}{s!}$$

$$= \frac{1}{e} \sum_{t=0}^{\infty} \sum_{s=0}^{n} \frac{\Delta^s 0^n}{s!(t-s)!} .$$

By interchanging the order of summations in the last expression, we obtain for the coefficient of e^{e^x-1}

$$\frac{1}{e} \sum_{s=0}^{n} \sum_{t=s}^{\infty} \frac{\Delta^s 0^s}{s!(t-s)!} = \frac{1}{e} \sum_{s=0}^{n} \frac{\Delta^s 0^n}{s!} e$$

where the e on the right end of the expression comes from summation over t.

Thus we have

$$\sum_{s=0}^{n} \frac{\Delta^s 0^n}{s!}$$

as the coefficient of $\frac{x^n}{n!}$ in the expansion of e^{e^x-1}.

Clearly any one of the expressions given in Theorem 1.2, 1.10, Theorem 1.8 could be used to compute the G's. However, an even more convenient computational procedure, which we actually used, will be discussed in the next chapter.

CHAPTER 2

ARITHMETIC PROPERTIES OF THE G's

In this chapter we will obtain a recurrence relation for the G's . We shall show that this relation may be used to define the G's , and we shall use this property in the compilation of Table I . The use of this relation in computing the G's gives rise to an array of numbers. We will denote these numbers by A_{mn} . We consider some arithmetic properties of these numbers and find some congruence relations for the G's and A's . Two determinantal forms for the G's are derived.

Theorem 2.1

$$G_{n+1} = \sum_{r=0}^{n} \binom{n}{r} G_r .$$

Proof:

$$\frac{1}{e} \sum_{s=0}^{\infty} \frac{(s+1)^n}{s!} = \frac{1}{e} \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{r=0}^{n} \binom{n}{r} s^r$$

$$= \frac{1}{e} \sum_{r=0}^{n} \binom{n}{r} \sum_{s=0}^{\infty} \frac{s^r}{s!}$$

$$= \sum_{r=0}^{n} \binom{n}{r} G_r .$$

First, we note that Theorem 2.1 can be written in the symbolic form

2.1 $$G_{n+1} = (G+1)^n$$

where, after expansion of the right hand member, powers of G are replaced by subscripts. We note that the recurrence relation 2.1 for the G's is similar to the recurrence relation

$$B_n = (B+1)^n$$

for the Bernoulli numbers.

Theorem 2.1 can also be proved as follows. Clearly G_{n+1} is the number of ways of putting $n+1$ points into classes. Specializing one of these points we see that this point may belong in a class with r others $(0 \le r \le n)$. These r points can be chosen in $\binom{n}{r}$ ways, and the remaining $n-r$ points can be disposed of in G_{n-r} ways. Hence

$$G_{n+1} = \sum_{r=0}^{n} \binom{n}{r} G_{n-r} .$$

This proof is due to Dubreil [21].

Theorem 2.2: The difference equation

$$\Delta^r K_1 = K_r$$

together with the initial condition

$$K_1 = 1$$

has as its unique solution $K_r = G_r$.

Proof:

$$K_{n+1} = E^n K_1 = (\Delta + 1)^n K_1 = \sum_{i=0}^{n} \binom{n}{i} \Delta^i K_1 .$$

Now, if these K's satisfy the conditions in Theorem 2.2 then

$$\sum_{i=0}^{n} \binom{n}{i} \Delta^i K_1 = \sum_{i=0}^{n} \binom{n}{i} K_i = (K+1)^i,$$

i.e. these numbers satisfy the recursion formula 2.1. Since we are given that $K_1 = 1$ Theorem 2.2 follows. This proof is due to Browne [16] and was given as the solution to a problem proposed by Becker [4]. Using Theorem 2.2, it is very easy to calculate the first few G's. The table given below shows how this is done. It is this method that is used in the construction of Table I .

1	1	2	5	15
2	3	7	10	
5	10	27		
15	37			
52				

The arrangement of the elements in Table I is slightly different from that just shown. The elements in any column of Table I are such that the sum $i+j$ in the expression $\Delta^i G_j$ is a constant for that column. In other words, the columns in Table I are the diagonal lines running from

upper right to lower left in the sample table given above: e.g. the elements of the fourth column in Table I, running from top to bottom are 5, 7, 10, 15. Table I was computed by the author.

The following theorems, 2.4-2.8, are due to Williams[52].

We need the following lemma in the proof of Theorem 2.4:

Lemma 2.1:

$$\sum_{s=0}^{n} \binom{n}{s}(-1)^{n-s} s = 0 \quad \text{for } n > 1.$$

Proof:

$$(x-1)^n = \sum_{s=0}^{n} \binom{n}{s} x^s (-1)^{n-s}.$$

Differentiation of this equation with respect to x yields

$$n(x-1)^{n-1} = \sum_{s=0}^{n} \binom{n}{s} s\, x^{s-1} (-1)^{n-s}.$$

Setting $x=1$ this yields

$$0 = \sum_{s=0}^{n} \binom{n}{s} s (-1)^{n-s}.$$

Theorem 2.4: For any prime p,

$$G_p \equiv 2 \pmod{p}.$$

Proof: By 1.10

$$G_p = \sum_{n=1}^{p} \frac{\Delta^n 0^p}{n!}.$$

Now

$$\frac{\Delta^1 0^p}{1!} = \frac{\Delta^p 0^p}{p!} = 1$$

and

$$\frac{\Delta^r 0^p}{r!} = \frac{(E-1)^r 0^p}{r!} = \frac{\sum_{s=0}^{r} \binom{r}{s}(-1)^{r-s} E^s 0^p}{r!}$$

$$= \frac{\sum_{s=0}^{r} \binom{r}{s}(-1)^{r-s} s^p}{r!}$$

But by Fermat's little theorem

$$s^p \equiv s \pmod{p}$$

Hence

$$\frac{\Delta^r 0^p}{r!} \equiv \frac{\sum_{s=0}^{r} \binom{r}{s}(-1)^{r-s} s}{r!} \pmod{p}.$$

Thus for $1 < r < p$ we have, by Lemma 2.1,

$$\frac{\Delta^r 0^p}{r!} \equiv 0 \pmod{p}.$$

And therefore

$$G_p = \sum_{r=1}^{p} \frac{\Delta^r 0^p}{r!} \equiv 1 + 0 + 0 + \cdots + 1 \equiv 2 \pmod{p}.$$

Theorem 2.4 may also be proven geometrically as follows. We consider p points (where p is a prime) arranged at the vertices of a regular polygon. We can represent any division of the p points into classes by joining the points by means of convex polygons. We next show that, except for the trivial cases in which all the points are joined or none of the points are joined, no rotation less than a complete revolution can bring any such configuration back

into itself.

Suppose we label the points around the polygon by the numbers 0, 1, 2, ... $p - 1$. Also, suppose that the rotation which carries the figure back onto itself is the one which carries 0 into r where $r < p$. Now r must have been a point in the original configuration and by the same rotation it is carried into $2r \pmod{p}$. Similarly, $2r$ is carried into $3r \pmod{p}$ and so on. That is, each point $nr \pmod{p}$ is carried into $(n + 1)r \pmod{p}$. Conversely, any point carried into a point $nr \pmod{p}$ must have been the point $(n - 1)r \pmod{p}$. Hence, some point $nr \pmod{p}$ must be carried into the point 0. But since we suppose that the rotation was not a complete revolution we also know that $n < p$. This implies $nr \equiv 0 \pmod{p}$ for some $n < p$. This is impossible because p is prime. Thus, any non-trivial configuration gives rise to p distinct configurations by rotating the configuration through

$$\frac{2\pi}{p}, \frac{4\pi}{p}, \frac{6\pi}{p}, \ldots \frac{(p-1)2\pi}{p}.$$

This last proof is due to Maranda and Moser [36], [37].

Before proceeding to the next theorem we need the following lemma:

Lemma 2.2. If

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} G_{n-r+1} = F(n,k)$$

and

$$\sum_{r=0}^{n-k} \binom{n-k}{r} G_{n-r} = H(n,k),$$

then

$$F(n,k) - F(n-1,k) = F(n,k+1)$$

and

$$H(n,k) - H(n-1,k) = H(n,k+1).$$

Proof:

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} G_{n-r+1} - \sum_{r=0}^{k} (-1)^r \binom{k}{r} G_{n-r}$$

$$= \sum_{r=0}^{k} (-1)^r \binom{k}{r} G_{n-r+1} - \sum_{r=1}^{k+1} (-1)^{r-1} \binom{k}{r-1} G_{n-r+1}$$

$$= (-1)^0 \binom{k}{0} G_{n-0+1} + \sum_{r=1}^{k} (-1)^r \binom{k}{r} G_{n-r+1}$$

$$+ \sum_{r=1}^{k} (-1)^r \binom{k}{r-1} G_{n-r+1} + (-1)^{k+1} \binom{k}{r} G_{n-k}.$$

But

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}.$$

hence

$$(-1)^0 \binom{k}{0} G_{n-0+1} + \sum_{r=1}^{k} (-1)^r \binom{k}{r} G_{n-r+1}$$

$$+ \sum_{r=1}^{k} (-1)^r \binom{k}{r-1} G_{n-r+1} + (-1)^{k+1} \binom{k}{r} G_{n-k}$$

$$= G_{n+1} + \sum_{r=1}^{k} (-1)^r \binom{k+1}{r} G_{n-r+1} + (-1)^{k+1} G_{n-k}$$

$$= \sum_{r=0}^{k} (-1)^r \binom{k+1}{r} G_{n-r+1}.$$

Thus,

$$F(n,k) - F(n-1,k) = F(n,k+1)$$

and similarly

$$H(n,k) - H(n-1,k) = H(n,k+1)$$

which completes the proof.

Theorem 2.5:

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} G_{n-r+1} = \sum_{r=0}^{n-k} \binom{n-k}{r} G_{n-r} \quad (0 \leq k \leq n).$$

Proof: For $k=0$, the theorem is true by Theorem 2.1. Hence, we may proceed by induction on k. We let

$$F(n,k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} G_{n-r+1}$$

and

$$H(n,k) = \sum_{r=0}^{n-k} \binom{n-k}{r} G_{n-r}.$$

and apply Lemma 2.2.

Theorem 2.6:

$$G_{p+n} \equiv G_n + G_{n+1} \pmod{p}.$$

Proof: When $n=0$ this reduces to Theorem 2.4. We proceed by induction over n. Now we write $p+n-1$ for n and p for k in Theorem 2.5 to obtain

$$G_{p+n} - G_n \equiv \sum_{r=0}^{n-1} \binom{n-1}{r} G_{p+n-1-r} \quad .$$

All other terms on the left side of the equation may be dropped in the congruence relation since they are divisible by p.

Now, assuming the theorem for all integers less than n we have

$$G_{p+n} - G_n \equiv \sum_{r=0}^{n-1} \binom{n-1}{r} G_{n-r+1} + \sum_{r=0}^{n-1} \binom{n-1}{r} G_{n-r} \pmod{p}$$
$$= \sum_{r=0}^{n} \binom{n}{r} G_{n-r} = G_{n+1}.$$

This theorem can be generalized to the following:

Theorem 2.7:

$$G_{kp^s+n} \equiv G^n (G+s)^k \pmod{p}.$$

Proof: For $s=0$ and all n the theorem is trivial. We proceed by induction on k and s. Assuming the theorem for some s and k and all n we have

$$G_{(k+1)p^s+n} = G_{kp^s+p^s+n} \equiv G^{p^s+n}(G+s)^k$$
$$\equiv G^n (G+s)^{k+1} \pmod{p}.$$

Hence, the theorem is true for all k. Now, assuming the theorem for all k, all n and some s, we let $k=p$ to obtain

$$G_{p^{s+1}+n} \equiv G^n (G+s)^p \equiv G_{p+n} + s^p G_n$$
$$\equiv G_{n+1} + G_n + s^p G_n$$
$$\equiv G_{n+1} + (s+1) G_n \pmod{p}.$$

Theorem 2.8:

$$G_{\sum_{i=0}^{n} a_i p^i} \equiv \prod_{i=0}^{n} (G+i)^{a_i}.$$

Proof: This follows at once from repeated application of Theorem 2.7 when the subscript of G is expressed as a polynomial in p.

Moser [39] gives theorems 2.9 and 2.10 below.

We define a double array of numbers A_{mn} as follows
$A_{mn} = A_{m-1,n-1} + A_{m-1,n}$ $(1<m\le n)$; $A_{11}=1$; $A_{1,m} = A_{m-1,m-1}$.
It follows from Theorem 2.2 that $A_{mm} = G_m$.
Table I gives $A_{m,n}$ for $(1\le m\le 25)$, $(1\le n\le 25)$.

Theorem 2.9:

$$G_{p+n} \equiv A_{2,n+2} \pmod{p}.$$

Proof:

$$G_{p+n} \equiv G_n + G_{n+1} = A_{n,n} + A_{n+1,n+1}$$

$$= A_{1,n+1} + A_{1,n+2} = A_{2,n+2} \pmod{p}.$$

Theorem 2.10:

$$G_{ap+b} \equiv A_{a+b+1,a+1} \pmod{p}.$$

Proof: For $a=1$ this theorem reduces to Theorem 2.9 so we proceed by induction on a. Thus we have

$$G_{ap+b} \equiv G_{(a-1)p+b} + G_{(a-1)p+b+1}$$

$$\equiv A_{a+b,a} + A_{a+b+1,a} = A_{a+b+1,a+1}.$$

Theorem 2.11: The sum of $\frac{p^p-1}{p-1}$ consecutive G's is divisible by p.

A proof of this theorem was communicated by Kaplansky to Williams and is given in Williams [52] . The proof involves the theory of difference equations in finite fields and will not be given here.

Theorem 2.12: The G's have a congruence period of length $\frac{p^p-1}{p-1}$.

Proof: By Theorem 2.11

$$\sum_{r=n}^{n+\frac{p^p-1}{p-1}-1} G_r \equiv \sum_{r=n+1}^{n+\frac{p^p-1}{p-1}} G_r \equiv 0 \pmod{p}.$$

Hence

$$G_n \equiv G_{n+\frac{p^p-1}{p-1}} .$$

Theorem 2.13:

$$G(G-1)(G-2)\cdots(G-n+1)=1.$$

Proof: We prove the theorem symbolically. We have

$$\sum_{i=0}^{\infty} \frac{G_i x^i}{i!} = e^{e^x-1} .$$

Now let

$$x = \log(1+u)$$

and obtain

$$\sum_{i=0}^{\infty} \frac{G_i[\log(1+u)]^i}{i!} = e^u$$

or

$$e^{G\log(1+u)} = e^u$$

so that

$$(1+u)^G = e^u .$$

Expanding the left hand side gives

$$\sum_{i=0}^{\infty} G(G-1)\cdots(G-i+1)\frac{u^i}{i!} = e^u.$$

Now, equating powers of u on both sides we have

$$G(G-1)\cdots(G-i+1)=1$$

We note that this relation can also be used to compute the G's recursively. Further, expansion of the product exhibits a relation between the Stirling numbers of the first kind and the G's.

J. Ginsburg [25] proves the following theorem which gives G_n in determinantal form.

Theorem 2.14:

$$G_{n+1} = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & -1 & 0 & 0 & \dots & 0 \\ 1 & 3 & 3 & 1 & -1 & 0 & \dots & 0 \\ 1 & 4 & 6 & 4 & 1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \binom{n}{5} & \dots & 1 \end{vmatrix} .$$

Proof: Consider the recursion formula in Theorem 2.1. for $n = 0, 1, 2 \dots, n$. This yields $n + 1$ equations in the unknowns $G_1, G_2, \dots G_{n+1}$. Solving these for G_{n+1} by determinants gives the required result.

Ginsburg [25] derives another determinantal expression for the G's . We need the following lemma.

Lemma 2.2:

If $\quad N = \sum_{i=1}^{\infty} A_i \frac{x^i}{i!} \quad$ then $\quad e^N = 1 + \sum_{i=1}^{\infty} B_i \frac{x^i}{i!}$

where

$$B_i = \begin{vmatrix} A_1 & -1 & \cdots & 0 & 0 \\ A_2 & A_1 & \cdots & 0 & 0 \\ A_3 & 2A_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & & \\ A_{i-1} & \binom{i-2}{1}A_{i-2} & \cdots & & -1 \\ A_i & \binom{i-1}{1}A_{i-1} & \cdots & & A_1 \end{vmatrix} .$$

Proof: If

$$y = e^N$$

then

$$\frac{dy}{dx} = y \sum_{i=0}^{\infty} A_{i+1} \frac{x^i}{i!} = \sum_{i=0}^{\infty} B_{i+1} \frac{x^i}{i!}$$

or
$$\left(1 + \sum_{i=1}^{\infty} B_i \frac{x^i}{i!}\right)\left(\sum_{i=0}^{\infty} A_{i+1} \frac{x^i}{i!}\right) = \sum_{i=0}^{\infty} B_{i+1} \frac{x^i}{i!}$$

which may be written symbolically as

$$e^{Bx} \cdot A e^{Ax} = B e^{Bx}.$$

Differentiating this n times with respect to x and setting $x = 0$ gives the recurrsion formula

2.2 $$B_{n+1} = A(B+A)^n.$$

Expanding 2.2 for $n = 0, 1, 2 \cdots n$ and replacing the exponents by subscripts after expansion we obtain

$$B_1 = A_1$$
$$B_2 = B_1 A_1 + A_2$$
$$B_3 = B_2 A_1 + 2B_1 A_2 + A_3$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The solution of these equations leads to the result.

Theorem 2.15:

$$G_n = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ \frac{1}{1!} & 1 & -2 & \cdots & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -(n-1) \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & 1 \end{vmatrix} .$$

Proof: We apply Lemma 2.2 to the equation

$$e^{e^x - 1} = e^{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$= 1 + x + \frac{x^2}{2!}\begin{vmatrix} 1 & -1 \\ \frac{1}{1!} & 1 \end{vmatrix} + \frac{x^3}{3!}\begin{vmatrix} 1 & -1 & 0 \\ \frac{1}{1!} & 1 & -2 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{1!} \end{vmatrix} + \cdots$$

CHAPTER 3

ANALYTIC PROPERTIES OF THE G's.

In this chapter, following L. Epstein[22] we generalize the function G_n to a function G_z, where z is a complex variable. As a special case of the complex variable we have the negative integers and we obtain an integral expression for G_{-n}. Maclaurin's expansion is used to express $G_{z+\delta}$ in a power series expansion in powers of δ. A general integral formula involving the G_z is obtained. We give a power series expansion for G_z with z pure imaginary and z complex. A few summation formulae involving G_z are given and the expansion with n replaced by the complex z is expressed in terms of sines and cosines. Two expressions involving differential operators are given to evaluate G_n. We conclude the chapter with two asymptotic expansions for G_n.

First, then, we define G_z for z complex.

Definition 3.1

$$G_z = \frac{1}{e}\sum_{t=0}^{\infty}\frac{t^z}{t!}.$$

This definition is meaningful because the series is convergent for all values of z as is seen by the ratio test, and, in fact, is an entire analytic function of z. For a real

negative integer, the series in Definition 3.1 converges very rapidly. Table IV for G_{-n} was evaluated from this series.

Theorem 3.1

$$G_{-1} = \frac{1}{e}\int_0^1 \frac{1}{x}(e^x - 1)\,dx .$$

Proof:

$$G_{-1} = \frac{1}{e}\sum_{t=1}^{\infty} \frac{1}{t\cdot t!}$$

but

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

hence

$$\frac{1}{e}\int_0^1 \frac{1}{x}(e^x - 1)\,dx = \frac{1}{e}\int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n(n!)}\Big|_{x=1} = \sum_{n=1}^{\infty} \frac{1}{n(n!)} = G_{-1} .$$

Similarly

$$G_{-2} = \frac{1}{e}\int_0^1 \frac{1}{x_1}\int_0^{x_1} (e^x - 1)\,dx\,dx_1 ,$$

and in general, we have

Theorem 3.2:

$$G_{-n} = \frac{1}{e}\int_0^1 \frac{1}{x_n}\int_0^{x_n} \frac{1}{x_{n-1}} \cdots \int_0^{x_2} \frac{1}{x_1}(e^{x_1} - 1)\,dx_1\,dx_2 \cdots dx_n .$$

Theorem 3.3:

$$G_{x+\xi} = \sum_{n=0}^{\infty}\left[\frac{1}{e}\sum_{t=0}^{\infty} \frac{t^x}{t!}(\ln t)^n\right]\frac{\xi^n}{n!}$$

Proof: Expanding $G_{x+\xi}$ in powers of ξ by Maclaurin's expansion we have

$$G_{x+\xi} = \sum_{n=0}^{\infty} G_x^{(n)} \frac{\xi^n}{n!} .$$

But

$$G_x^{(n)} = \frac{d^n}{dx^n} \frac{1}{e} \sum_{t=1}^{\infty} \frac{t^x}{t!}.$$

Now the series obtained from differentiating term by term, i.e.

$$\frac{1}{e} \sum_{t=0}^{\infty} \frac{t^x}{t!} (\ln t)^n$$

converges uniformly in x for x in any bounded region. For, using the ratio test

$$\left| \frac{u_{t+1}}{u_t} \right| = \left| \frac{(t+1)^x}{t+1} \frac{[\ln(t+1)]^n}{[\ln(t)]^n} \right| \to 0$$

as $t \to \infty$. Hence

$$G_x^{(n)} = \frac{1}{e} \sum_{t=0}^{\infty} \frac{t^x}{t!} (\ln t)^n.$$

If we let

$$p_n = \sum_{t=1}^{\infty} \frac{(\ln t)^n}{n!}$$

we have

Theorem 3.4:

$$G_\xi = \frac{1}{e} \sum_{n=0}^{\infty} \frac{p_n}{n!} \xi^n.$$

Proof: We set $x = 0$ in Theorem 3.3.

Table III is given by Epstein [22] and gives p_m from $m=0$ to 15.

Theorem 3.5:

$$\sum_{n=1}^{\infty} \frac{p_n}{n!} = 1.$$

Proof: We set $\xi = 1$ in Theorem 3.4.

Theorem 3.6:

$$\int_a^b G_x\, f(x)\, dx = \frac{1}{e}\sum_{t=0}^{\infty} \frac{1}{t!}\int_a^b f(x) e^{x \ln t}\, dx.$$

for $b > a > 0$ and $f(x)$ integrable in $a \leq x \leq b$.

Proof: Let $z = x$ in Definition 3.1 and multiply both sides of the equation by $f(x)$. Since the series converges uniformly we may integrate the right hand side term by term.

Theorem 3.7:

$$G_{x+iy} = \frac{1}{e}\left[\sum_{t=0}^{\infty} \frac{t^x \cos(\ln t)y}{t!} + i\sum_{t=0}^{\infty} \frac{t^x \sin(\ln t)y}{t!}\right].$$

Proof: By definition

$$G_{x+iy} = \frac{1}{e}\sum_{t=0}^{\infty} \frac{t^{x+iy}}{t!}.$$

Separation of the series into real and imaginary parts completes the proof.

Theorem 3.8:

$$G_{iy} = \frac{1}{e}\left[\sum_{t=0}^{\infty} \frac{\cos(\ln t)y}{t!} + i\sum_{t=0}^{\infty} \frac{\sin(\ln t)y}{t!}\right].$$

Proof: We let $x = 0$ in Theorem 3.7.

Theorem 3.9:

$$\sum_{n=0}^{\infty} \frac{G_{2n}}{(2n)!} = \frac{1}{2e}\left(e^{e} + e^{\frac{1}{e}}\right).$$

Proof:

$$\sum_{n=0}^{\infty} \frac{G_n}{n!} = \frac{1}{e} e^{e}$$

and

$$\sum_{n=0}^{\infty} \frac{G_n (-1)^n}{n!} = \frac{1}{e} e^{\frac{1}{e}}.$$

But

3.1 $$\sum_{n=0}^{\infty} \frac{G_{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{G_{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{G_n}{n!} = \frac{1}{e} e^{e}$$

and

3.2 $$\sum_{n=0}^{\infty} \frac{G_{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{G_{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{G_n (-1)^n}{n!} = \frac{1}{e} e^{\frac{1}{e}}$$

Now we add 3.1 and 3.2.

Theorem 3.10:

$$\sum_{n=0}^{\infty} \frac{G_{2n+1}}{(2n+1)!} = \frac{1}{2e} \left(e^{e} - e^{\frac{1}{e}} \right).$$

Proof: We subtract 3.2 from 3.1.

We note here that the expressions corresponding to 3.1 and 3.2 and the Theorem 3.9 are given incorrectly by Epstein[2].

Theorem 3.11:

$$\frac{1}{e} e^{e^{x+iy}} = \frac{1}{e} e^{e^x \cos y}[\cos(e^x \sin y) + i \sin(e^x \sin y)].$$

Proof:

$$e^{e^{x+iy}} = e^{e^x \cos y} \cdot e^{ie^x \sin y}$$

$$= e^{e^x \cos y}[\cos(e^x \sin y) + i \sin(e^x \sin y)].$$

Aitken [1] states, without proof,

Theorem 3.12: If $g(x) = f[u(x)]$, then in the expansion of $\frac{d^i g}{dx^i}$ the constant part of the coefficient of $\frac{d^n f}{du^n}$ is ${}_n S_i$.

Proof:

3.4 $$\frac{d^n g}{dx^n} = \sum_{i=1}^{n} a_i \frac{d^i f}{du^i}$$

where a_i is independent of g . So we choose g conveniently as

$$g = e^{\omega u}.$$

Now

$$\frac{d^i e^{wu}}{du^i} = w^i e^{wu}.$$

Hence

$$\frac{d^n e^{wu}}{dx^n} = \sum_{i=1}^{n} a_i w^i e^{wu}.$$

Differentiating both sides with respect to w , i times, and then letting $w=0$ we have

3.5 $$\frac{1}{i!} \frac{d^i}{dw^i} \left[e^{-wu} \frac{d^n e^{wu}}{dx^n} \right]_{w=0} = a_i$$

But

$$\frac{d^i}{dx^i} \varphi(x+s) = \frac{d^i}{ds^i} \varphi(x+s)$$

so

$$\frac{d^i}{dx^i} \varphi(x) = \left[\frac{d^i}{dx^i} \varphi(x+t) \right]_{t=0} = \left[\frac{d^i}{dt^i} \varphi(x+t) \right]_{t=0}.$$

Hence

$$\frac{d^n}{dx^n} e^{wu(x)} = \left[\frac{d^n}{dh^n} e^{wu(x+h)} \right]_{h=0}.$$

Thus, from 3.5

$$a_i = \frac{1}{i!} \frac{d^i}{dw^i} \left[e^{-wu} \left(\frac{d^n}{dh^n} e^{wu(x+h)} \right)_{h=0} \right]_{w=0}$$

$$= \frac{1}{i!} \left[\frac{d^{i+n}}{dw^i dh^n} e^{wu(x+h) - wu(x)} \right]_{h=0,\, w=0}$$

$$= \frac{1}{i!} \left[\frac{d^{i+n}}{dw^i dh^n} e^{w \underset{h}{\Delta} u(x)} \right]_{h=0,\, w=0}$$

$$= \frac{1}{i!} \left[\frac{d^n}{dh^n} \left(\underset{h}{\Delta} u(x) \right)^i \right]_{h=0}.$$

Now, if we let

$$u(x) = e^{x}$$

we have

$$[\underset{h}{\Delta}\, u(x)]^{i} = [\underset{h}{\Delta}\, e^{x}]^{i} = [e^{x+h} - e^{x}]^{i}$$

$$= e^{xi}[e^{h}-1]^{i} = e^{xi}\sum_{r=0}^{i}\binom{i}{r}(-1)^{r} e^{h(i-r)}$$

so that

$$a_i = \frac{1}{i!}\Big[(i-r)^{n} e^{xi}\sum_{r=0}^{i}\binom{i}{r}(-1)^{r} e^{h(i-r)}\Big]_{h=0}$$

$$= \frac{e^{xi}}{i!}\sum_{r=0}^{i}\binom{i}{r}(i-r)^{n}(-1)^{r}.$$

If now we let $x=0$ we have

3.6 $$a_i = \frac{1}{i!}\sum_{r=0}^{i}\binom{i}{r}(i-r)^{n}(-1)^{r}.$$

But

3.7 $$\sum_{r=0}^{i}\binom{i}{r}(i-r)^{n}(-1)^{r} = {}_{i}S_{n}$$

for $$x^{n} = \sum_{i=1}^{n}(\Delta^{i} x^{n})_{x=0}\,\frac{(x)_i}{i!}$$

analagous to

$$x^{n} = \sum_{i=1}^{n}(D^{i} x^{n})\Big|_{x=0}\,\frac{x^{i}}{i!}$$

By definition

$$\left(\frac{\Delta^{i} x^{n}}{i!}\right)_{x=0} = {}_{i}S_{n}$$

and

$$\Delta^i = \sum_{s=0}^{i} (-1)^s \binom{i}{s} E^{i-s}$$

Therefore

$$\left[\frac{\Delta^i x^n}{i!}\right]_{x=0} = \frac{1}{i!}\sum_{s=0}^{i} (-1)^s \binom{i}{s} (i-s)^n .$$

Thus, from 3.4 and the choices $f = e^{w^u}, w = e^x$ for f and w we have

$$3.8 \quad \left[\frac{d^n e^{e^x}}{dx^n}\right]_{x=0} = \sum_{i=0}^{n} {}_iS_n \left[\frac{d^i e^{e^x}}{de^{x\,i}}\right]_{x=0} = e\sum_{i=0}^{n} {}_iS_n$$

$$= e\, C_n .$$

We note that this is exactly the result we obtained in Theorem 1.6.

Aitken[1] also gives a relationship involving the operator xD to evaluate the C's. He states, without proof, that the coefficients of x^iD^i in the expansion of $(xD)^n$ are the same as those of $\frac{d^if}{dw^i}$ in the expansion of $\frac{d^ng}{dx^n}$.

Thus

$$3.9 \quad (xD)^n = \sum_{i=0}^{n} {}_iS_n\, x^iD^i$$

Using the operand e^{x-1} in 3.9 we have:

$$3.10 \quad \left[(xD)^n e^{x-1}\right]_{x=1} = \left[\sum_{i=0}^{n} {}_iS_n\, x^iD^i e^{x-1}\right]_{x=1} = C_n$$

Comparing 3.10 with 3.8 we have

$$[D^n e^{e^x-1}]_{x=0} = [(xD)^n e^{x-1}]_{x=1}$$

$$= [\{(x+1)D\}^n e^x]_{x=0} .$$

It is clearly of interest to have asymptotic formulas for G_n . Epstein [23] gives the following theorem:

Theorem 3.13:

$$G_n \sim \left[\frac{n e^{\frac{1}{\ln n}}}{\ln n} \right]^n .$$

We give here an outline of Epstein's proof, but point out by a numerical example that the result is very poor at n = 25.

By use of the Euler-Maclaurin sum formula we convert the sum expression for the G's .

$$e G_n = \sum_{x=0}^{\infty} \frac{x^n}{\Gamma(x+1)}$$

into an asymptotically equivalent integral. This gives

$$3.11 \quad e G_n = \int_0^{\infty} \frac{x^n dx}{\Gamma(x+1)} - S_n$$

where

$$3.12 \quad S_n = \sum_{m=1}^{\infty} (-1)^m \frac{B_m'}{(2m)!} \left[\left\{ \frac{x^n}{\Gamma(x+1)} \right\}^{(2m-1)} \Bigg|_{x=\infty} - \left\{ \frac{x^n}{\Gamma(x+1)} \right\}^{(2m-1)} \Bigg|_{x=0} \right]$$

and B_m' are Bernoulli's numbers. The S_n is reduced to the

form

$$3.13 \quad S_n = -\sum_{m=m_0}^{\infty} (-1)^m \frac{B_m}{2m} \cdot a_{2m-n}$$

where

$$m_0 = \frac{n+1}{2} \qquad , n \quad \text{odd}$$

and

$$m_0 = \frac{n+2}{2} \qquad , n \quad \text{even}$$

and the a_n appear in the expansion

$$\frac{1}{\Gamma(x)} = \sum_{k=1}^{\infty} a_k x^k .$$

Tables of coefficients a_1 to a_{23} are given in Bourquet[1] . Next we show that the series 3.13 may be terminated after any number of terms with error less than the next term of the series. Then the integral

$$\int_0^{\infty} \frac{x^n}{\Gamma(x+1)} dx$$

is reduced to the form

$$\int_0^{\infty} \frac{x^n}{\Gamma(x+1)} dx \sim \sqrt{2\pi} \frac{\alpha_n^{n+1}}{\Gamma(\alpha_n+1)} \cdot \frac{1}{\sqrt{n+\alpha_n^2 \Psi'(\alpha_n+1)}}$$

where

$$\Psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

and α_n is the solution of the equation

$$\alpha_n = \frac{n}{\Psi(\alpha_n+1)} .$$

Thus

$$G_n \sim \frac{\sqrt{2\pi}}{e} \frac{\alpha_n^{\,n+1}}{\Gamma(\alpha_n+1)} \frac{1}{\sqrt{n+\alpha_n^2 \Psi'(\alpha_n+1)}} + \frac{1}{e} \sum_{m=m_0} (-1)^m \frac{B_m}{2m} a_{2m-n} .$$

We show that if we are interested in an asymptotic formula only, all but the first term can be dropped. For large n further approximations are applied to give

$$G_n \sim \frac{\alpha_n^{\,n-\beta_n} \, e^{\beta_n - 1}}{\sqrt{\ln \beta_n}}$$

where β_n is the solution of

$$\frac{n}{\ln(\beta_n+1)} = \beta_n$$

β_n can be found by the scheme

$$\beta_n = \cfrac{n}{\ln\left(1 + \cfrac{n}{\ln\left(1 + \cfrac{n}{\ln n}\right.}\right.} \quad .$$

For n large

$$\beta_n \sim \frac{n}{\ln n - \ln(\ln \alpha_n)}$$

and

$$\beta_n \sim \frac{n}{\ln n}$$

giving

$$G_n \sim \left[\frac{n e^{\frac{1}{\ln n}}}{\ln n} \right]^n$$

to complete the proof.

As stated earlier, this result is unsatisfactory as it gives

$$G_{25} = 4.24 \times 10^{25}$$

which is too large by a factor of about 10^7 .

We next present an asymptotic formula for G_n due to Wyman [53], together with an outline of his proof.

Theorem 3.14: Let R be the real solution of $Re^R = n$. Then

$$G_n \sim \frac{e^{n(R+\frac{1}{R}-1)-1}}{\sqrt{R+1}}$$

Proof: By 3.8 we have

$$G_n = \left[\frac{d^n}{dt^n} e^{e^t - 1}\right]_{t=0}.$$

Hence, by Cauchy's theorem

$$G_n = \frac{n!}{2\pi i}\int_C \frac{e^{e^z-1}}{z^n}\,dz$$

where C is a closed contour containing the origin. Choosing C to be the circle $|z| = R$ with $Re^R = n$ we have

$$3.14 \quad G_n = \frac{n!}{2\pi e R^n}\int_{-\pi}^{\pi} e^{e^{Re^{i\theta}}} e^{-in\theta}\,d\theta.$$

We are interested in an asymptotic formula as $n \to \infty$ (and hence $R \to \infty$). This enables us to replace the limits in the last integral by $\pm\epsilon$, $\epsilon > 0$, the remaining part of the integral being of smaller order. We can then use an expansion for $e^{Re^{i\theta}}$ in powers of θ. To derive the later we note that

$$\frac{d}{d\theta}\left(e^{Re^{i\theta}}\right)\Big]_{\theta=0} = e^R(i)(R)$$

$$\frac{d^2}{d\theta^2}\left(e^{Re^{i\theta}}\right)\Big]_{\theta=0} = e^R(-i)(R^2+R)$$

$$\frac{d^3}{d\theta^3}\left(e^{Re^{i\theta}}\right)\Big]_{\theta=0} = e^R(-i)(R^3+3R^2+R)$$

and in general

$$\frac{d^n}{d\theta^n}\left(e^{Re^{i\theta}}\right)\Big]_{\theta=0} = e^R(i^n)P_n(R)$$

where $P_n(R)$ is a polynomial of degree n with leading coefficient 1. Thus we obtain

$$3.15 \quad e^{Re^{i\theta}} = e^R\Big[iR\theta - (R^2+R)\frac{\theta^2}{2!} - i(R^3+3R^2+R)\frac{\theta^3}{3!} + \cdots\Big].$$

It may also be shown that if 3.15 is used in 3.14 the terms of 3.15 in θ^3 and beyond contribute to the result only a factor which rapidly approaches 1. Thus from 3.14 and 3.15 we obtain

$$3.16 \quad \tilde{\omega}_n \sim \frac{n!}{2\pi e R^n}\int_{-\epsilon}^{\epsilon} e^{i(Re^R-n)}\cdot e^{-e^R(R^2+R)\frac{\theta^2}{2!}}\,d\theta.$$

Noting that $Re^R = n$ and making the substitution

$$3.17 \quad \theta = \sqrt{\frac{2}{e^R(R^2+R)}}\;\phi$$

we find

3.18
$$G_n \sim \frac{n!\, e^{e^R} \sqrt{2}}{2\pi e R^n \sqrt{e^R(R^2+R)}} \int_{-\epsilon\sqrt{\frac{e^R(R^2+R)}{2}}}^{\epsilon\sqrt{\frac{e^R(R^2+R)}{2}}} e^{-\varphi^2}\, d\varphi.$$

As $n \to \infty$, $R \to \infty$ and the last integral approaches π .

Hence

3.19
$$G_n \sim \frac{n!\, e^{e^R}}{\sqrt{2}\, e R^n \sqrt{e^R(R^2+R)}} .$$

Various forms of 3.19 can be obtained by using $Re^R = n$

and Stirling's formula

3.20
$$n! \sim n^n e^{-n} \sqrt{2\pi n} .$$

To obtain the form given in the statement of the theorem, note that

3.21 $e^R = \frac{n}{R}$

and

3.22 $R^n = n^n e^{-Rn}$.

Use of 3.19, 3.20, 3.21, 3.22, yields, after some simplification, the required result.

For $n=25$ Theorem 3.14 yields $G_{25} \doteq 4.7 \times 10^{18}$.
The correct value, to three figures of accuracy, is $G_{25} \doteq 4.64 \times 10^{18}$.

CHAPTER 4

REPORT ON MISCELLANEOUS PAPERS

In this chapter the results of several authors on studies pertaining to the C's will be presented briefly in chronological order.

Boole[11], in 1880, proposed the following problem. Show that

$$e^{e^t} = e\{1 + (e^{\Delta}0)t + (e^{\Delta}0^2)\frac{t^2}{1\cdot 2} + \cdots\}$$

where
$$e^{\Delta}0^i = \sum_{n=0}^{\infty} \frac{\Delta^n 0^i}{n!} .$$

We require the following lemma for the solution of this problem.

Lemma 4.1: If $\phi(x)$ is an analytic function, regular in any neighborhood of the origin then

$$\phi(e^t) = \phi(E)e^{0\cdot t}$$

Proof:

$$\phi(e^t) = \sum_{i=0}^{\infty} a_i e^{it}$$

$$= \sum_{i=0}^{\infty} a_i E^i e^{0\cdot t} = \phi(E)e^{0\cdot t}.$$

Now we can prove

Theorem 4.1:

$$e^{e^t} = e\{1+(e^{\Delta}0)t + (e^{\Delta}0^2)\frac{t^2}{1\cdot 2} + \cdots\}$$

Proof: Let

$$\varphi(x) = \frac{1}{e}e^{x}.$$

Hence, by Lemma 4.1

$$\frac{1}{e}e^{e^t} = \varphi(e^t) = \frac{1}{e}e^{E}e^{0\cdot t}$$

$$= \frac{1}{e}e^{1+\Delta}\sum_{i=0}^{\infty}\frac{0^i t^i}{i!}$$

$$= e^{\Delta}\sum_{i=0}^{\infty}\frac{0^i t^i}{i!} = \sum_{i=0}^{\infty}(e^{\Delta}0^i)\frac{t^i}{i!}.$$

We may carry this a step farther to obtain

$$\frac{1}{e}e^{e^t} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{\Delta^j 0^i}{j!}\right)\frac{t^i}{i!} = \sum_{i=0}^{\infty}G_i\frac{t^i}{i!}$$

and so obtain the G's as the divided differences of zero.

In 1885 Cesàro[19] gave the earliest explicit solution for the G's.

By letting

$$(G+1) = v$$

Cesàro finds

$$e^{(G+1)x} = e^{vx} = \sum_{i=0}^{\infty}\frac{v^i x^i}{i!} = \sum_{i=0}^{\infty}\frac{(G+1)^i x^i}{i!}$$

$$= \sum_{i=0}^{\infty}\frac{G^{i+1}x^i}{i!} = G\sum_{i=0}^{\infty}\frac{G^i x^i}{i!} = Ge^{Gx} = \frac{d}{dx}e^{Gx}.$$

Thus e^{Gx} satisfies the differential equation

$$y\,e^{x} = \frac{dy}{dx}$$

whose general solution is evidently

$$y = e^{e^{x}-c}$$

By letting $x = 0$ we find $c = 1$.

Then

$$e^{e^{x}-1} = e^{Gx} = \sum_{k=0}^{\infty} G_k \frac{x^k}{k!}$$

Cèsaro's paper ends with the remarkable formula

$$G_p = \frac{2}{\pi e}\int_0^{\pi} e^{e^{\cos\theta}\cos(\sin\theta)} \sin\{e^{\cos\theta}\sin(\sin\theta)\}\sin p\theta\, d\theta .$$

We note that this formula does not agree with the corresponding formula derived in Chapter 3.

Anderegg[1][5] proved

$$G_n = \begin{vmatrix} 1 & -\binom{n-1}{0} & -\binom{n-1}{1} & \cdots\cdots & \binom{n-1}{n-2} \\ 1 & 1 & -\binom{n-2}{0} & \cdots\cdots & \binom{n-2}{n-3} \\ 1 & 0 & 1 & \cdots\cdots & \binom{n-3}{n-4} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ 1 & 0 & 0 & \cdots\cdots & \binom{1}{0} \\ 1 & 0 & 0 & \cdots\cdots & 1 \end{vmatrix}$$

and also gave the values of G_1 to G_{10} .

Epstein[24] studied the arithmetical properties of the G's. His works were published in 1905.

In 1905, Krug[34] published some studies on the arithmetical properties of the G's.

In 1908, Bromwich[15] proposed the following problem.

If

$$S_r = \sum_{n=1}^{\infty} \frac{n^r}{n!}$$

show S_r is an integral multiple of e, and in particular

$$S_1 = e, \quad S_2 = 2e, \quad S_3 = 5e$$
$$S_4 = 15e, \quad S_5 = 52e, \quad S_6 = 203e$$
$$S_7 = 877e, \quad S_8 = 4,140e$$

We see that

$$\frac{S_r}{e}$$

is identical with G_r.

In 1908, Hardy [27] proposed the following problems.

(i) Sum the series $\sum_{n=0}^{\infty} P_r(n)\frac{x^n}{n!}$ where $P_r(n)$ is a polynomial of degree r in n. Note that we can express

in the form

$$A_0 + A_1 n + A_2 n(n-1) + \cdots + A_r (n)(n-1)\cdots(n-r+1)$$

and

$$\sum_{n=0}^{\infty} P_r(n)\frac{x^n}{n!} = A_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + A_1 \sum_{x=1}^{\infty} \frac{x^n}{(n-1)!} + \cdots + A_r \sum_{x=r}^{\infty} \frac{x^n}{(n-r)!}$$

$$= (A_0 + A_1 x + A_2 x^2 + \cdots + A_r x^r) e^x.$$

(ii) Show that

$$\sum_{n=1}^{\infty} \frac{n^3}{n!} x^n = (x + 3x^2 + x^3) e^x, \quad \sum_{n=1}^{\infty} \frac{n^4}{n!} x^n = (x + 7x^2 + 6x^3 + x^4) e^x,$$

and that if

$$S_n = 1^3 + 2^3 + \cdots + n^3$$

then

$$\sum_{n=1}^{\infty} S_n \frac{x^n}{n!} = \frac{1}{4}(4x + 14x^2 + 8x^3 + x^4) e^x.$$

In particular the last series is equal to zero when $x = -2$.

Prove that

$$\sum_{n=0}^{\infty} \frac{n}{n!} = e, \quad \sum_{n=0}^{\infty} \frac{n^2}{n!} = 2e, \quad \sum_{n=0}^{\infty} \frac{n^3}{n!} = 5e$$

and that

$$\sum_{n=0}^{\infty} \frac{n^k}{n!}$$

where k is any positive integer, is a positive integral multiple of e.

Schwatt [43], in 1924, by studying properties of the differential operator

$$\left(x\frac{d}{dx}\right)^n$$

found

$$G_n = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^{\alpha} \binom{k}{\alpha} \alpha^n$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \left[\binom{k}{1} 1^n - \binom{k}{2} 2^n + \cdots + (-1)^{k-1} \binom{k}{k} k^n \right]$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \left[\frac{1^n}{1!} \frac{1}{(k-1)!} - \frac{2^n}{2!} \frac{1}{(k-2)!} + \cdots (-1)^{k-1} \frac{k^n}{k!} \right].$$

In 1925, Whitworth [51] showed that the total number of ways in which n different objects can be distributed into 1, 2, 3, ..., n indifferent parcels is G_n . He proved further

$$4.1 \qquad G_n = n! \sum_{t=1}^{n} N_t$$

where N_t is the number of t-partitions of n different things, and showed that N_t is $n!$ times the coefficient of x^n in the expansion of

$$\frac{(e^x-1)^t}{t!} .$$

In 1928 Ginsburg [26] gave a brief history of the Stirling numbers of the first and second kinds. We restrict ourselves here to the results relating to Stirling numbers of the second kind $_m\mathcal{J}_n$. These numbers may be generated by the following scheme.

We let

$$\varphi_0 = e^x, \quad \varphi_n = (x\varphi_{n-1})'.$$

Thus

$$\varphi_1 = e^x(1)$$
$$\varphi_2 = e^x(x+1)$$
$$\varphi_3 = e^x(1x^2+3x+1)$$
$$\varphi_4 = e^x(x^3+6x^2+7x+1)$$
$$\cdots\cdots$$
$$\varphi_n = e^x({}_nS_n x^{n-1} + {}_{n-1}S_n x^{n-2} + \cdots + {}_2S_n x + {}_1S_n)$$

We note in passing that C_n is just $\dfrac{\varphi_n(1)}{e}$.

The same coefficients appear in the expansion of x^n in factorials. Thus

$$x = x$$
$$x^2 = x(x-1) + x$$
$$x^3 = x(x-1)(x-2) + 3x(x-1) + x$$
$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$
$$\cdots\cdots$$
$$x^n = \sum_{i=0}^{n-1} {}_{n-i}S_n\,(x)_{n-i}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1).$$

Tables of Stirling numbers of the second kind were given by Boole[11] and Cayley[18]. Our Table II is an extension of Cayley's.

Ginsburg concludes with the following formulae which are given without proof.

$$\frac{1+2x}{(1-x)^6} = {}_1S_3 + {}_2S_4\, x + {}_3S_5\, x^2 + \cdots + {}_nS_{n+2}\, x^{n-1} + \cdots$$

$$\frac{1+8x+6x^2}{(1-x)^7} = {}_1S_4 + {}_2S_5\, x + {}_3S_6\, x^2 + \cdots + {}_nS_{n+3}\, x^{n-1} + \cdots$$

$$\frac{1+22x+55x^2+24x^3}{(1-x)^9} = \sum_{i=0}^{\infty} {}_iS_{i+4}\, x^{i-1}.$$

$${}_{n-1}S_{n+1} = \binom{n+2}{4} + 2\binom{n+1}{4}.$$

$${}_{n-1}S_{n+2} = \binom{n+4}{6} + 8\binom{n+3}{6} + 6\binom{n+2}{6}.$$

$${}_{n-1}S_{n+3} = \binom{n+6}{8} + 22\binom{n+5}{8} + 58\binom{n+4}{8} + 24\binom{n+3}{8}.$$

In 1931, Chiellini [20] gave results on $\sum_{n=0}^{\infty} \frac{n^r}{n!}$ for integral r. Expanding the series in the inverse factorial series

$$\frac{n^r}{n!} \sum_{k=1}^{r} \frac{b_{rk}}{(n-k)!}$$

where

$$b_{rk} = \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \alpha^r$$

$$b_{rk} = k\, b_{r-1,k} + b_{r-1,k-1}$$

so that

$$b_{r1} = 1^{r-1}$$

$$b_{r2} + b_{r1} = 2^{r-1}$$

$$b_{r3} + b_{r2} + \frac{b_{r1}}{2!} = \frac{3^{r-1}}{2!}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$b_{rr} + b_{r,r-1} + \frac{b_{r,r-2}}{2!} + \cdots \frac{b_{r1}}{(r-1)!} = \frac{r^{r-1}}{(r-1)!}$$

we find

$$\sum_{n=1}^{\infty} \frac{n^r}{n!} = e \sum_{k=1}^{r} b_{rk} = b_r e$$

Hence b_r is identical to C_r. Chiellini includes a table of b_r for integral r $(r = 1, 2 \cdots n)$. This contains the errors

$$4,140 = b_8 \neq 4,138 .$$

$$21,147 = b_9 \neq 22,147 .$$

In 1933, Touchard[46] gave some arithmetic properties of the G's.

In 1933, Broggi[14] gave some results on $\sum_{n=1}^{\infty} \frac{n^h}{n!}$ for positive integral h. He used the classical Sterling expansion

$$\prod_{p=0}^{n} \frac{1}{(x+p)} = \sum_{s=0}^{\infty} (-1)^s \frac{C_n^s}{x^{n+s}}$$

Where

$$C_n^s = \frac{1}{n!} \sum_{r=0}^{n-1} (-1)^r (n-r)^{n+s} \binom{n}{r}$$

so that the C_n^s are what Nielson[41] calls the Stirling numbers of the second kind, and designates C_{n+1}^s. Then from the known properties of these numbers, he demonstrates

$$G_n = 1 + \frac{1(n-1)^n}{(n-1)!} + \frac{\left(1-\frac{1}{1!}\right)(n-2)^n}{(n-2)!} + \dots + \frac{\left(1-\frac{1}{1!}+\frac{1}{2!}+\cdots(-1)^n\right)\frac{1}{(n-2)!}}{1!},$$

$$G_n = C_n^0 + C_{n-1}^1 + C_{n-2}^2 + \dots + C_1^{n-1}.$$

and proves C_r^{r-k} is identical with Chiellini's b_{rk}.

Finally he derives the asymptotic series

$$\frac{1}{e}\int_0^1 t^{x-1} e^t \, dt = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{G_n}{x^{n+1}}.$$

Bell[5],[6],[7],[8],[9] contributed several papers on the G's. He showed

$$\xi_n^{(1)} = G_n = \sum_{s=1}^{n} \frac{1}{(s-1)!}\left[\sum_{r=0}^{s-1} (-1)^r \binom{s-1}{r} (s-r)^{n-1}\right] \quad (n>0)$$

and discussed some of the arithmetic properties of these numbers G_n. Further applications to the theory of numbers were also treated in the second and third of these papers. In the second paper he demonstrated

$$4.2 \qquad G_n = \sum_{p=1}^{n} \frac{\Delta^p 0^n}{p!}$$

where $\Delta^p 0^n$ are the "differences of zero" discussed by various writers [44].

Equation 4.2 was used, with tables of differences of zero, to find G_1 to G_{20}. The latest paper contains some interesting generalizations of the G's. It also gives an interesting interpretation of the significance of the G's in combinatorial analysis.

The following problem appears in Whittaker and Watson's text [50].

If

$$F_{a,n}(x) = \sum_{m=0}^{\infty} \frac{(m+a)^n}{m!} x^m$$

show that

$$F_{a,n}(x) = \left\{\frac{d^n}{dt^n}\left(e^{at+xe^t}\right)\right\}_{t=0} = e^x P_n(x,a)$$

where $P_n(x,a)$ is a polynomial of degree n in x, and deduce that

$$P_{n+1}(x,a) = (x+a)P_n(x,a) + x\frac{d}{dx}P_n(x,a).$$

We note that

$$F_{0,n}(1) = e\,G_n.$$

In 1939, Dubriel[21] showed that the number of equivalence relations, G_n, on n elements satisfies the recursion formula

$$G_{n+1} = (G+1)^n.$$

In 1941, Browne[6] proposed the following problem.

Show that the difference equation

$$\Delta^k N_1 = N_k, \quad N_1 = 1$$

defines the sequence

1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, ...

whose k^{th} term is $f^k(0)$, where

$$f(x) = e^{e^x - 1}.$$

Browne gives

$G_{30} =$ 864 740 014 511 809 332 450 147.

$G_{35} =$ 281 600 203 019 560 266 563 340 426 570.

In 1948, Birkoff[10] showed that the function

$$e^{e^x - 1}$$

generates the number of equivalence relations among n elements.

In 1951, Lambeck[35] gave an alternative proof connecting the combinatorial definition of G_n with the generating function.

In 1951, Westwick[49] considered the series

$$\sum_{p=0}^{\infty} \frac{n^p}{n!}$$

We begin with the series

$$\sum_{n=0}^{\infty} u_n \frac{x^n}{n!}$$

where u_n is a polynomial of degree p in n . The polynomial is written in the form

$$\sum_{i=0}^{p} a_i (n)_i$$

where the coefficients $a_0, a_1, \ldots a_p$ are independent of n. By re-arrangement the sum of the series is obtained as

$$e \sum_{i=0}^{p} a_i x^i$$

Taking the particular case in which $x = 1$, and $u_n = n^p$, the sum of this series is, since $a_0 = 0$,

$$e \sum_{i=0}^{p} a_i$$

where the a's can be found by solving in succession the equations

$$1 = a_1,$$
$$\frac{2^p}{2!} = a_1 + a_2$$
$$\frac{3^p}{3!} = \frac{a_1}{2!} + a_2 + a_3$$
$$\frac{4^p}{4!} = \frac{a_1}{3!} + \frac{a_2}{2!} + a_3 + a_4$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

This leads to

$$a_1 = 1,$$
$$a_2 = 2^{p-1} - 1$$

and generally,

$$a_r = \sum_{s=0}^{r-1} \frac{(-1)^s (r-s)^p}{s!\,(r-s)!}$$

Thus

$$\sum_{r=1}^{p} a_r = \left\{1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \cdots + \frac{p^p}{p!}\right\} - \left\{1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \cdots + \frac{(p-1)^p}{(p-1)!}\right\}$$
$$+ \frac{1}{2!}\left\{1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \cdots \frac{(p-2)^p}{(p-2)!}\right\} + \cdots \frac{(-1)^{p-1}}{(p-1)!}.$$

This is essentially the second form given by Mendelsohn [38].

He then gave, without proof, the following algorithm for the sum of the a's. We form the table

1

1 2

2 3 5

5 7 10 15

15 20 27 37 52

52 67 87 114 151 203

203 255 322 409 523 674 877

. . . .

The formation is as follows:

The first element of any horizontal row is the same as the last element of the preceding row;

The m^{th} element ($m>1$) of any row is obtained by the addition of the $(m-1)^{th}$ element of that row and the $(m-1)^{th}$ element of the preceding row.

Then the required sum is the last element in the p^{th} row, and the sum of the series is this number multiplied by e .

We note that

$$\sum_{n=1}^{p} a_n = \mathcal{B}_p .$$

In 1952, Maranda [36] gave an alternative proof for the explicit formula for $\mathcal{B}_n$.

Burger [17], in 1952, proposed the following problem.

Show that the Bernoulli numbers B_n, defined by

4.3 $$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

satisfy

4.4 $$B_n = \sum_{\nu=0}^{n} \frac{(-1)^\nu \nu!}{1+\nu} \, {}_\nu S_n$$

where ${}_\nu S_n$ are the Stirling numbers of the second kind given in Definition 1.1 .

We define the polynomials $G_n(x)$ by

$$G_n(x) = e^{-x}\left(x\frac{d}{dx}\right)^n e^x.$$

It may be shown that

$$G_{n+1}(x) = x[G_n(x) + G_n'(x)]$$

and

$$G_n(x) = \sum_{\nu=1}^{n} {}_\nu S_n \, x^\nu.$$

Also

$$e^{x(e^z-1)} = \sum_{n=0}^{\infty} \frac{G_n(x)}{n!} z^n.$$

Thus we have

$$G_n(1) = C_n$$

Another relation derived in the solution of the problem is

$$\sum_{n=0}^{\infty} \frac{{}_{\nu}S_n \, x^n}{n!} = \frac{1}{\nu!}\left(e^x - 1\right)^{\nu}.$$

Finally, we note that, using the explicit expression for ${}_{\nu}S_n$ given in (4.3) yields

$$B_n = \sum_{\nu=0}^{n} \sum_{\mu=0}^{n} \frac{(-1)^{\nu}\binom{\mu}{\nu}}{1+\mu} \, \nu^n .$$

Knopp[32] considerd the problem of proving

$$g^{p+1} = (g+1)^p$$

starting at

$$\frac{1}{e}\sum_{n=1}^{\infty} \frac{n^p}{n!} = g^p .$$

He gives [32] two asymptotic expressions involving the g^p.

$$\operatorname{Log} g^p = n\left[\operatorname{Log} \frac{n}{e \operatorname{Log} n} + \epsilon_n\right], \text{ where } \epsilon_n \to 0$$

$$\frac{g^n}{n!} = \left(\frac{1+\eta_n}{\operatorname{Log} n}\right)^n, \text{ where } \eta_n \to 0$$

Thus, we see that his g^p is the same as our G_p. In the second asymptotic expression for the g^p he does not state the manner in which η_n approaches zero so the formula is of no use in computing the g^p.

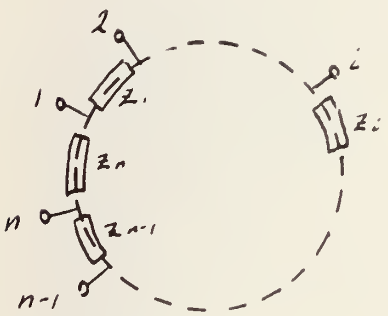
Vadnal[47] considered some properties of the sum of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

CHAPTER 5

APPLICATIONS

In this chapter we shall first consider the way in which the G's can be used to express the number of measurable impedances of an n-terminal network. Secondly, we shall show how the G's arise in the solution of a problem in statistics.

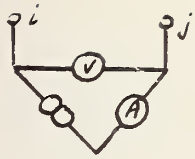
Riordan [42] treats the n-terminal network problem as follows: An n-terminal passive network is represented at the left. z_i represent various impedances and the little circles labeled 1, 2, 3, ... n represent terminals connected to the ends of the impedances as shown. To make impedance measurements on the network we need a power source, ammeter and voltmeter which are represented by the symbols below.



 source,  voltmeter,  ammeter.

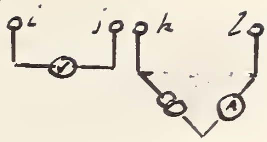
The following arrangements will be used in making impedance measurements on the n-terminal network.

(i)



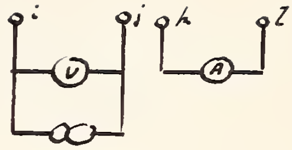
Driving point
impedance (D)

(ii)



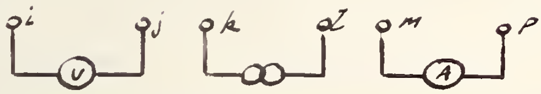
(Open circuit)
Transfer Impedances (T)

(iii)



(Short circuit)

(iv)



Generalized Transfer Impedances (U)

Any one of the arrangements (i), (ii), (iii), (iv) will be used at one time to make an impedance measurement on the n-terminal network. The contacts numbered i, j, k, l, m, p are all between 1 and n and are connected to the corresponding contacts of the n-terminal network. They are subject to the conditions

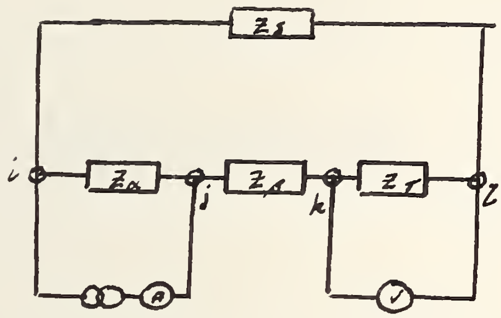
$$i \neq j, \quad k \neq l, \quad m \neq p.$$

Furthermore, i, j, k, l, m, p are not to be equated pairwise in any way whereby any one of the arrangements (i), (ii), (iii), (iv) is transformed into another one of the arrangements.

For any one of the arrangements connected to the n-terminal network, as outlined above, we define the impedance

as the ratio of the readings $\frac{V}{I}$.

If we make an open circuit transfer impedance measurement on an n-terminal passive network we will obtain a circuit such as is shown below



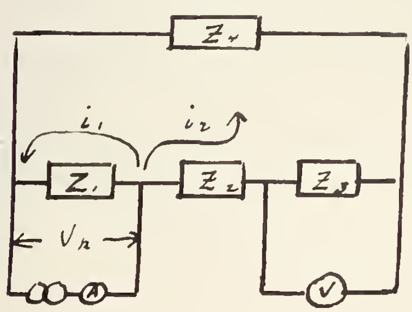
Again, i, j, k, l, represent numbers between 1 and n.

$Z_\alpha, Z_\beta, Z_\gamma, Z_\delta$, represent the total impedances between the various pairs of terminals.

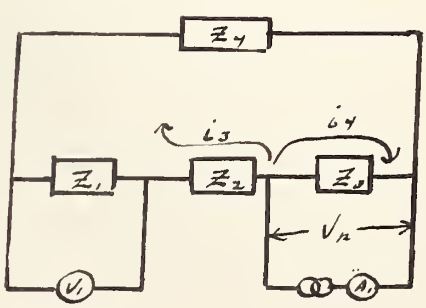
Theorem 5.1: In the T class circuits any impedance is unaltered by an interchange of voltmeter (ammeter) with associated source and ammeter (voltmeter).

Proof: We prove that $\frac{V}{I} = \frac{V_1}{I_1}$ in the circuits (v) and (vi) below.

(v)



(vi)



The effective impedance across V_h in (v) is

$$\frac{1}{\dfrac{1}{Z_2+Z_3+Z_4} + \dfrac{1}{Z_1}} = \frac{Z_1(Z_2+Z_3+Z_4)}{Z_1+Z_2+Z_3+Z_4}$$

Hence

(5.1) $$I = \frac{V_h(Z_1+Z_2+Z_3+Z_4)}{Z_1(Z_2+Z_3+Z_4)} .$$

Also

(5.2) $$V = i_2 Z_3 .$$

The effective impedance across V_k in (vi) is

$$\frac{1}{\dfrac{1}{Z_1+Z_2+Z_4} + \dfrac{1}{Z_3}} = \frac{Z_3(Z_1+Z_2+Z_4)}{Z_1+Z_2+Z_3+Z_4} .$$

Hence

(5.3) $$I_1 = \frac{V_k(Z_1+Z_2+Z_3+Z_4)}{Z_3(Z_1+Z_2+Z_4)}$$

Also

(5.4) $$V_1 = i_3 Z_1 .$$

But

(5.5) $$i_1 Z_1 = i_2(Z_2 + Z_3 + Z_4) = i_4 Z_3 = i_3(Z_2 + Z_1 + Z_4).$$

Hence, by (5.1) and (5.2)

(5.6) $$\frac{V}{I} = \frac{i_2 Z_3 Z_1 (Z_2 + Z_3 + Z_4)}{V_k (Z_1 + Z_2 + Z_3 + Z_4)} .$$

By (5.3) and (5.4)

(5.7) $$\frac{V_1}{I_1} = \frac{i_3 Z_1 Z_3 (Z_1 + Z_2 + Z_4)}{V_k (Z_1 + Z_2 + Z_3 + Z_4)} .$$

By (5.5)

$$i_3 = i_2 \frac{(Z_2 + Z_3 + Z_4)}{(Z_2 + Z_1 + Z_4)} .$$

Hence, by (5.7)

(5.8) $$\frac{V_1}{I_1} = \frac{i_2 (Z_2 + Z_3 + Z_4)}{(Z_2 + Z_1 + Z_4)} \frac{Z_1 Z_3 (Z_1 + Z_2 + Z_4)}{V_k (Z_1 + Z_2 + Z_3 + Z_4)}$$

$$= \frac{i_2 Z_1 Z_3 (Z_2 + Z_3 + Z_4)}{V_k (Z_1 + Z_2 + Z_3 + Z_4)} .$$

and $\frac{V}{I} = \frac{V_1}{I_1}$

by (5.8) and (5.6) .

If now we let $T^{o}_{x,n}$ and $T^{s}_{x,n}$ be the numbers of open-circuit and short-circuit transfer impedances, respectively, measurable when short circuits have been placed across the n terminals in all possible ways to leave x unshorted terminals, we have the following theorem.

Theorem 5.2:

$$T^{o}_{x+1,n} = T^{s}_{x,n}$$

Proof: Given x separate terminals, there are the same number of short-circuit transfer impedances measurable as open-circuit impedances. A short-circuit transfer impedance measurement shorts out a pair of terminals to leave only $x-1$ separate terminals.

Now we let $D_{x,n}$ represent the number of driving-point impedances measurable for all possible mergings of n terminals such that x are left unshorted. We let D_n represent the total number of driving-point transfer impedances measurable for an n-terminal network. We make similar definitions for $T_{x,n}$, $U_{x,n}$, and T_n, U_n, corresponding to transfer and generalized impedance measurement, respectively. The following theorem then follows immediately.

Theorem 5.3: Each of D_n, T_n and U_n can be found by summing $D_{x,n}$, $T_{x,n}$, and $U_{x,n}$ respectively over x from $x=2$ to $x=n$. i.e.

$$D_n = \sum_{x=2}^{n} D_{x,n}$$
$$T_n = \sum_{x=2}^{n} T_{x,n}$$
$$U_n = \sum_{x=2}^{n} U_{x,n}$$

Now, we let d_x, t_x, and u_x be the number of driving-point, transfer, and generalized transfer impedances, respectively, measurable for x unshorted terminals. The following theorem results.

Theorem 5.4:

$$\begin{pmatrix} D_n \\ T_n \\ U_n \end{pmatrix} = \sum_{x=2}^{n} \begin{pmatrix} d_x \\ t_x \\ u_x \end{pmatrix} {}_xS_n$$

where ${}_xS_n$ are Stirling numbers of the second kind defined in Definition 1.1.

Proof: $D_{x,m}$ is the product of two factors: the number of such impedances measurable for x terminals, which is independent of n, and the number of ways n terminals may be merged to leave x separate, which is independent of the impedance class. ${}_xS_n$ is the number of ways n terminals may be merged to leave x separate (c.f. the discussion leading up to 1.10).

It is through the ${}_xS_n$ that the G's enter into the

problem. We recall that ${}_xJ_n = \frac{\Delta^x 0^n}{x!}$. (Definition 1.1)

Theorem 5.5:

$$d_x = \frac{1}{2}(x)_2 .$$

Proof: A driving point impedance may be measured between every pair of terminals; hence d_x is the number of combinations of x things taken two at a time, that is

$$d_x = \binom{x}{2} = \frac{1}{2}(x)_2 .$$

Theorem 5.6:

$$t_x = \frac{1}{8}[4(x)_3 + (x)_4].$$

Proof: For a given pair of driving terminals, there are $\binom{x}{2} - 1$ measurable open-circuit transfer impedances since a voltmeter can be connected to every pair of the x terminals except the driving pair; hence, multiplying by the number of driving terminals and by the factor one-half to eliminate reciprocity duplicates (c.f. Theorem 5.1) we have the result.

By Theorem 5.2 this serves for enumeration of both open-circuit and short-circuit transfer impedances.

Theorem 5.7:

$$w_x = \frac{1}{8}[20(x)_4 + 10(x)_5 + (x)_6].$$

Proof: Considering, for the generalized transfer impedances, a

fixed source and an ammeter in a fixed (non-source) position, the voltmeter may be connected across $\binom{x}{2}$ pairs of terminals when x terminals are available; one of these pairs is the source pair measuring a short-circuit transfer impedance which must be excluded. Hence, remembering that reciprocity theorem duplicates are eliminated:

$$U_{x,n} = 2\left[\binom{x}{2}-1\right] T^{s}_{x,n}.$$

Hence, by Theorem 5.2

$$U_{x,n} = 2\left[\binom{x}{2}-1\right] T^{o}_{x+1,n}$$

and by Theorem 5.4

$$U_{x,n} = 2\left[\binom{x}{2}-1\right] t_{x+1} \left({}_{x+1}S_n\right).$$

Degrading x by unity, we obtain the result.

Lemma 5.1: The generating identity for the function

$$\sum_{x=0}^{n} a^{x}\, {}_{x}S_{n}$$

is

(5.9) $$e^{a(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{x=0}^{n} a^x\, {}_{x}S_n .$$

Proof:

$$e^{a(e^t-1)} = \frac{1}{e^a}\, e^{ae^t} = \frac{1}{e^a}\sum_{s=0}^{\infty}\frac{(ae^t)^s}{s!} = \frac{1}{e^a}\sum_{s=0}^{\infty}\frac{a^s}{s!}\sum_{r=0}^{\infty}\frac{(ts)^r}{r!}$$

$$= \frac{1}{e^a}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{s=0}^{\infty}\frac{s^r}{s!}\,a^s = \frac{1}{e^a}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{s=0}^{\infty}\sum_{u=0}^{r}\frac{\Delta^u 0^r}{u!(s-u)!}\,a^s$$

$$= \frac{1}{e^a}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{s=0}^{\infty}\sum_{u=0}^{r}\frac{\Delta^u 0^r}{u!(s-u)!}\,a^{s-u}\,a^u$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{u=0}^{r}\frac{\Delta^u 0^r}{u!}\,a^u.$$

But $\frac{\Delta^u 0^r}{u!} = {}_uS_r$, hence the result.

Lemma 5.2:

$$(e^t-1)^s e^{(e^t-1)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{x=0}^{n}(x)_s\,({}_xS_n).$$

Proof: We differentiate (5.9) s times with respect to a and set $a = 1$.

Theorem 5.8:

$$e^{tD} = \tfrac{1}{2}(e^t-1)^2 e^{(e^t-1)},$$
$$e^{tT} = \tfrac{1}{8}\left[4(e^t-1)^3 + (e^t-1)^4\right]e^{(e^t-1)},$$
$$e^{tU} = \tfrac{1}{8}\left[20(e^t-1)^4 + 10(e^t-1)^5 + (e^t-1)^6\right]e^{(e^t-1)}.$$

Proof: Substituting the value obtained from Theorem 5.5 for d_x into Theorem 5.4 we have

(5.9) $D_n = \sum_{x=2}^{n}\frac{1}{2}(x)_2\;{}_xS_n$

or

$$(5.10) \quad \sum_{x=0}^{n} (x)_2 \, {}_xS_n = 2D_n$$

Now we let $s=0$ in Lemma 5.2 and substitute $2D_n$ for $\sum_{x=0}^{n} (x)_2 ({}_xS_n)$ by (5.10). Thus

$$\frac{1}{2}(e^t-1)^2 e^{(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_n$$

or, symbolically,

$$e^{tD} = \frac{1}{2}(e^t-1)^2 e^{(e^t-1)}.$$

Similarly, we prove the second and third equations of the theorem.

Further expansion of the brackets in the equations of Theorem 5.8 gives the following

$$(5.11) \quad \begin{aligned} e^{tD} &= \tfrac{1}{2}(e^{2t}-2e^t+1)e^{e^t-1} \\ e^{tT} &= \tfrac{1}{8}(e^{4t}-6e^{2t}+8e^t-3)e^{e^t-1} \\ e^{tU} &= \tfrac{1}{8}(e^{6t}+4e^{5t}-15e^{4t}+35e^{2t} \\ &\qquad -36e^t+11)e^{e^t-1}. \end{aligned}$$

Theorem 5.9:

$$D_n = \frac{1}{2}[(G+2)^n - 2(G+1)^n + G_n],$$

$$T_n = \frac{1}{8}[(G+4)^n - 6(G+2)^n + 8(G+1)^n - 3G_n]$$

$$U_n = \frac{1}{8}[(G+6)^n + 4(G+5)^n - 15(G+4)^n + 35(G+2)^n$$
$$-36(G+1)^n + 11G_n].$$

Proof: We write e^{e^t-1} as e^{Gt} in equations 5.11, and pass from generating relations to coefficient relations. Thus

$$\sum_{n=0}^{\infty} \frac{t^n D_n}{n!} e^{tD} = \frac{1}{2}(e^{2t} - 2e^t + 1)e^{Gt} =$$

$$= \frac{1}{2}(e^{(2+G)t} - 2e^{(1+G)t} + e^{Gt})$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(G+2)^n t^n}{n!} - 2\sum_{n=0}^{\infty} \frac{(G+1)^n t^n}{n!} + \sum_{n=0}^{\infty} \frac{G_n t^n}{n!}\right].$$

or

$$D_n = \frac{1}{2}\left[(G+2)^n - 2(G+1)^n + G_n\right]$$

Similarly, we derive the equations for T_n and U_n .

Theorem 5.10:

$$G_{n+1} = (G+1)^n$$
$$G_{n+2} = (G+1)^n + (G+2)^n$$
$$G_{n+3} = (G+1)^n + 3(G+2)^n + (G+3)^n$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$G_{n+m} = \sum_{x=1}^{m} (G+x)^n {}_x S_m$$

Proof: We differentiate the generating identity

$$e^{Gt} = e^{e^t-1}$$

of the G's with respect to t repeatedly. Then we pass from the generating relations to coefficient relations.

We now define Stirling numbers of the first kind.

Definition 5.1: $S_{x,m}$ is a Stirling number of the first kind where

$$S_{x,m+1} = S_{x-1,m} - m S_{x,m}$$

subject to the conditions

$$S_{m,m} = 1; \quad S_{x,m} = 0 \text{ for } x > m; \quad S_{0,m} = 0 \text{ for } m > 0.$$

We then have the following lemma relating the Stirling numbers of the first and second kinds. (c.f. Definition 1.1 for Stirling numbers of the second kind).

Lemma 5.3:

If

$$a_m = \sum_{x=1}^{m} b_x \, ({}_x S_m)$$

then

$$b_m = \sum_{x=1}^{m} a_x \, S_{x,m}$$

We do not prove this lemma.

Theorem 5.11:

$$D_n = \frac{1}{2}[G_{n+2} - 3G_{n+1} + G_n]$$

$$T_n = \frac{1}{8}[G_{n+4} - 6G_{n+3} + 5G_{n+2} + 8G_{n+1} - 3G_n]$$

$$U_n = \frac{1}{8}[G_{n+6} - 11G_{n+5} + 3G_{n+4} + 5G_{n+3} - 56G_{n+2} - 5G_{n+1} + 11G_n].$$

Proof: We apply Lemma 5.3 to the last equation of Theorem 5.10 to obtain

$$(5.12)\quad (G+m)^n = \sum_{x=1}^{m} G_{n+x} \, S_{x,m}.$$

We then compute the first few Stirling numbers of the first kind by the recursion relation of Definition 5.1. Using

5.12, we are then able to calculate $(G+i)^n$ for various i and so transform the equations of Theorem 5.9 into those of Theorem 5.11.

For numerical checks, it is convenient to note the simplest congruences for the three numbers. These follow from the congruence for the G's, which is

$$G_{p+n} \equiv G_{n+1} + G_n \pmod{p}$$

where p is a rational prime greater than 2. (c.f. Theorem 2.6).

Theorem 5.12:

$$D_{p+n} \equiv D_{n+1} + D_n \pmod{p}$$
$$T_{p+n} \equiv T_{n+1} + T_n \pmod{p}$$
$$U_{p+n} \equiv U_{n+1} + U_n \pmod{p}.$$

Proof: Since, by the equations of Theorem 5.11, each of the impedance numbers is a linear function of the G's, the result follows.

We now consider the manner in which the G's appear in the solution to a problem in statistics.

Following Weatherburn [48] consider a variate x with probability density $f(x)$. The expected value of e^{tx} is, by definition,

$$M(t) = \int e^{tx} f(x)\, dx$$

(the integration being over the whole range of x). If the integral has meaning for a certain range of values of t we may integrate term by term to get

$$M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \cdots$$

where

$$\mu_r' = \int x^r f(x)\, dx$$

is the moment of order r about the mean.

Definition:

$$M(t) = \int e^{tx} f(x)\, dx$$

will be called the moment generating function of the distribution, with probability density $f(x)$, about the origin.

Definition: $\int x^r f(x)\, dx$ is the moment of order r about the mean of the distribution with probability density $f(x)$.

Theorem: The n^{th} moment of Poisson's distribution with mean 1 is C_n.

Proof: The probability density of Poisson's distribution is

$$\frac{m^x e^{-m}}{x!}$$

So the moment generating function is

$$\int e^{tx} \frac{m^x e^{-m}}{x!}\, dx$$

where the integral is a Stieltjes integral.

Now,

$$\int e^{tx} \frac{m^x e^{-m}}{x!} dx = e^{-m} \int \sum_{p=0}^{\infty} \frac{(tx)^p}{p!} \frac{m^x}{x!} dx$$

$$= e^{-m} \sum_{p=0}^{\infty} \frac{t^p}{p!} \int \frac{x^p m^x}{x!} dx$$

$$= e^{-m} \sum_{p=0}^{\infty} \frac{t^p}{p!} \left[\frac{1^p m}{1!} + \frac{2^p m^2}{2!} + \frac{3^p m^3}{3!} + \cdots \right]$$

$$= e^{-m} \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \frac{(ts)^p m^s}{p!\, s!}$$

$$= e^{-m} \sum_{s=0}^{\infty} \frac{e^{ts} m^s}{s!}$$

$$= e^{-m} e^{me^t}$$

$$= e^{m(e^t - 1)}$$

and the proof is complete.

TABLE I: G_n UP TO n = 25

G_0	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
1	1	2	5	15	52	203	877	4 140
	2	3	7	20	67	255	1 080	5 017
		5	10	27	87	322	1 335	6 097
			15	37	114	409	1 657	7 432
				52	151	523	2 066	9 089
					203	674	2 589	11 155
						877	3 263	13 744
							4 140	17 007
								21 147

G_9	G_{10}	G_{11}	G_{12}
21 147	115 975	678 570	4 213 597
25 287	137 122	794 545	4 892 167
30 304	162 409	931 667	5 686 712
36 401	192 713	1 094 076	6 618 379
43 833	229 114	1 286 789	7 712 455
52 922	272 947	1 515 903	8 999 244
64 077	325 869	1 788 850	10 515 147
77 821	389 946	2 114 719	12 303 997
94 828	467 767	2 504 665	14 418 716
115 975	562 595	2 972 432	16 923 381
	678 570	2 535 027	19 895 813
		4 213 597	23 430 840
			27 644 437

G_{13}	G_{14}	G_{15}	G_{16}
27 644 437	190 899 322	1 382 958 545	10 480 142 147
31 858 034	218 543 759	1 573 857 867	11 863 100 692
36 750 201	250 401 793	1 792 401 626	13 436 958 559
42 436 913	287 151 994	2 042 803 419	15 229 360 185
49 055 292	329 588 907	2 329 955 413	17 272 163 604
56 767 747	378 644 199	2 659 544 320	19 602 119 017
65 766 991	435 411 946	3 038 188 519	22 261 663 337
76 282 138	501 178 937	3 473 600 465	25 299 851 856
88 586 135	577 461 075	3 974 779 402	28 773 452 321
103 004 851	666 047 210	4 552 240 477	32 748 231 723
119 928 232	769 052 061	5 218 287 687	37 300 472 200
139 824 045	888 980 293	5 987 339 748	42 518 759 887
163 254 885	1 028 804 338	6 876 320 041	48 506 099 635
190 899 322	1 192 059 223	7 905 124 379	55 382 419 676
	1 382 958 545	9 097 183 602	63 287 544 055
		10 480 142 147	72 384 727 657
			82 864 869 804

G_{17}

82 864 869 804
93 345 011 951
105 208 112 643
118 645 071 202
133 874 431 387
151 146 594 991
170 748 714 008
193 010 377 345
218 310 229 201
247 083 681 522
279 831 913 245
317 132 385 445
359 651 145 332
408 157 244 976
463 539 664 643
526 827 208 698
599 211 936 355
682 076 806 159

G_{18}

682 076 806 159
764 941 675 963
858 286 687 914
963 494 800 557
1 082 139 871 759
1 216 014 303 146
1 367 160 898 137
1 537 909 612 145
1 730 919 989 490
1 949 230 218 691
2 196 313 900 213
2 476 145 813 458
2 793 278 198 903
3 152 929 344 235
3 561 086 589 202
4 024 626 253 845
4 551 453 462 543
5 150 665 398 898
5 832 742 205 057

G_{19}

5 832 742 205 057
6 514 819 011 216
7 279 760 687 179
8 138 047 375 093
9 101 542 175 650
10 183 682 047 409
11 399 696 350 555
12 766 857 248 692
14 304 766 860 837
16 035 686 850 327
17 984 917 069 018
20 181 230 969 231
22 657 376 782 689
25 450 654 981 592
28 603 584 325 827
32 146 670 915 029
36 189 297 168 874
40 740 750 631 417
45 891 416 030 315
51 724 158 235 372

G_{20}

51 724 158 235 372
57 556 900 440 429
64 071 719 451 645
71 351 480 138 824
79 489 527 513 917
88 591 069 689 567
98 774 751 736 976
110 174 448 087 531
122 941 305 336 223
137 246 072 197 060
153 281 759 047 387
171 266 676 116 405
191 447 907 085 636
214 105 283 868 325
239 555 938 849 917
268 159 523 175 744
300 324 194 090 773
336 513 491 259 647
377 254 241 891 064
423 145 657 921 379
474 869 816 156 751

G_{21}

474 869 816 156 751
526 593 974 392 123
584 150 874 832 552
648 222 594 284 197
719 574 074 423 021
799 063 601 936 938
887 654 671 626 505
986 429 423 363 481
1 096 603 871 451 012
1 219 545 176 787 235
1 356 791 248 984 295
1 510 073 008 031 682
1 681 339 684 148 087
1 872 787 591 233 723
2 086 892 875 102 048
2 326 448 813 951 965
2 594 608 337 127 709
2 894 932 531 218 482
3 231 446 022 478 129
3 608 700 264 369 193
4 031 845 922 290 572
4 506 715 738 447 323

G_{22}

4 506 715 738 447 323
4 981 585 554 604 074
5 508 179 528 996 197
6 092 330 403 828 749
6 740 552 998 112 946
7 460 127 072 535 967
8 259 190 674 472 905
9 146 845 346 099 410
10 133 274 769 462 891
11 229 878 640 913 903
12 449 423 817 701 138
13 806 215 066 685 433
15 316 288 074 717 115
16 997 627 758 865 202
18 870 415 350 098 925
20 957 308 225 200 973
23 283 757 039 152 938
25 878 365 376 280 647
28 773 297 907 499 129
32 004 743 929 977 258
35 613 444 194 346 451
39 645 290 116 637 023
44 152 005 855 084 346

G_{23}

44 152 005 855 084 346
48 658 721 593 531 669
53 640 307 148 135 743
59 148 486 677 131 940
65 240 817 080 960 689
71 981 370 079 073 635
79 441 497 151 609 602
87 700 687 826 082 507
96 847 533 172 181 917
106 980 807 941 644 808
118 210 686 582 558 711
130 660 110 400 259 849
144 466 325 466 945 282
159 782 613 541 662 397
176 780 241 300 527 599
195 650 656 650 626 524
216 607 964 875 827 497
239 891 721 914 980 435
265 770 087 291 261 082
294 543 385 198 760 211
326 548 129 128 737 469
362 161 573 323 083 920
401 806 863 439 720 943
445 958 869 294 805 289

G_{24}

445 958 869 294 805 289
490 110 875 149 889 635
538 769 596 743 421 304
592 409 903 891 557 047
651 558 390 568 688 987
716 799 207 649 649 676
788 780 577 728 723 311
868 222 074 880 332 913
955 922 762 706 415 420
1 052 770 295 878 597 337
1 159 751 103 820 242 145
1 277 961 790 402 800 856
1 408 621 900 803 060 705
1 553 088 226 270 005 987
1 712 870 839 811 668 384
1 889 651 081 112 195 983
2 085 301 737 762 822 507
2 301 909 702 638 650 004
2 541 801 424 553 630 439
2 807 571 511 844 891 521
3 102 114 897 043 651 732
3 428 663 026 172 389 201
3 790 824 599 495 473 121
4 192 631 462 935 194 064
4 638 590 332 229 999 353

G_{25}

4 638 590 332 229 999 353

Table II: Stirling Numbers of the Second Kind

m \ n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511
3			1	6	25	90	301	966	3 025	9 330
4				1	10	65	350	1 701	7 770	34 105
5					1	15	140	1 050	6 951	42 525
6						1	21	266	2 646	22 827
7							1	28	462	5 880
8								1	36	750
9									1	45
10										1

m \ n	11	12	13	14
1	1	1	1	1
2	1 023	2 047	4 095	8 191
3	28 501	86 526	261 625	788 970
4	145 750	611 501	2 532 530	10 391 745
5	246 730	1 379 400	7 508 501	40 075 035
6	179 487	1 323 652	9 321 312	63 436 373
7	63 987	627 396	5 715 424	49 329 280
8	11 880	159 027	1 899 612	20 912 320
9	1 155	22 275	359 502	5 135 130
10	55	1 705	39 325	752 752
11	1	66	2 431	66 066
12		1	78	3 367
13			1	91
14				1

m \ n	15	16	17
1	1	1	1
2	16 383	32 767	65 535
3	2 375 101	7 141 686	21 457 825
4	42 355 950	171 798 901	694 337 290
5	210 766 920	1 096 190 550	5 652 751 651
6	420 693 273	2 734 926 558	17 505 749 898
7	408 741 333	3 281 882 604	25 708 104 786
8	216 627 840	2 141 764 053	20 415 995 028
9	67 128 490	820 784 250	9 528 822 303
10	12 662 650	193 754 990	2 758 334 150
11	1 479 478	28 936 908	512 060 978
12	106 470	2 757 118	62 022 324
13	4 550	165 620	4 910 178
14	105	6 020	249 900
15	1	120	7 820
16		1	136
17			1

m \ n	18	19
1	1	1
2	131 071	262 143
3	64 439 010	193 448 101
4	2 798 806 985	11 259 666 950
5	28 958 095 545	147 589 284 710
6	110 687 251 039	693 081 601 779
7	197 462 483 400	1 492 924 634 839
8	189 036 065 010	1 709 751 003 480
9	106 175 395 755	1 144 614 626 805
10	37 112 163 803	477 297 033 785
11	8 391 004 908	129 413 217 791
12	1 256 328 866	23 466 951 300
13	125 854 638	2 892 439 160
14	8 408 778	243 577 530
15	367 200	13 916 778
16	9 996	527 136
17	153	12 597
18	1	171
19		1

m \ n	20	21
1	1	1
2	524 287	1 048 575
3	580 606 446	1 742 343 625
4	45 232 115 901	181 509 070 050
5	749 206 090 500	3 791 262 568 401
6	4 306 078 895 384	26 585 679 462 804
7	11 143 554 045 652	82 310 957 214 948
8	15 170 932 662 679	132 511 015 347 084
9	12 011 282 644 725	123 272 476 465 204
10	5 917 584 964 655	71 187 132 291 275
11	1 900 842 429 486	26 826 851 689 001
12	411 016 633 391	6 833 042 030 178
13	61 068 660 380	1 204 909 218 331
14	6 302 524 580	149 304 004 500
15	452 329 200	13 087 462 580
16	22 350 954	809 944 464
17	741 285	34 952 799
18	15 675	1 023 435
19	190	19 285
20	1	210
21		1

m \ n	22	23
1	1	1
2	2 097 151	4 194 303
3	5 228 079 450	15 686 335 501
4	727 778 623 825	2 916 342 574 750
5	19 137 821 912 055	96 416 888 184 100
6	163 305 339 345 225	998 969 857 983 405
7	602 762 379 967 440	4 382 641 999 117 305
8	1 142 399 079 991 620	9 741 955 019 900 400
9	1 241 963 303 533 920	12 320 068 811 796 900
10	835 143 799 377 954	9 593 401 297 313 460
11	366 282 500 870 286	4 864 251 308 951 100
12	108 823 356 051 137	1 672 162 773 483 930
13	22 496 861 868 481	401 282 560 341 390
14	3 295 165 281 331	68 629 175 807 115
15	345 615 943 200	8 479 404 429 331
16	26 046 574 004	762 361 127 264
17	1 404 142 047	49 916 988 803
18	53 374 629	2 364 885 369
19	1 389 850	79 781 779
20	23 485	1 859 550
21	231	28 336
22	1	253
23		1

m \ n	24
1	1
2	8 388 607
3	47 063 200 806
4	11 681 056 634 501
5	485 000 783 495 250
6	6 090 236 036 084 530
7	31 677 463 851 804 540
8	82 318 282 158 320 505
9	120 622 574 326 072 500
10	108 254 081 784 931 500
11	63 100 165 695 775 560
12	24 930 204 590 758 260
13	6 888 836 057 922 000
14	1 362 091 021 641 000
15	195 820 242 247 080
16	20 677 182 465 555
17	1 610 949 936 915
18	92 484 925 445
19	3 880 739 170
20	116 972 779
21	2 454 606
22	33 902
23	276
24	1

m \ n	25
1	1
2	16 777 215
3	141 197 991 025
4	46 771 289 738 810
5	2 436 684 974 110 751
6	37 026 417 000 002 430
7	227 832 482 998 716 310
8	690 223 721 118 368 580
9	1 167 921 451 092 973 005
10	1 203 163 392 175 387 500
11	802 355 904 438 462 660
12	362 262 620 784 874 680
13	114 485 073 343 744 260
14	25 958 110 360 896 000
15	4 299 394 655 347 200
16	526 655 161 695 960
17	48 063 331 393 100
18	3 275 678 594 925
19	166 218 969 675
20	6 220 194 750
21	168 519 505
22	3 200 450
23	40 250
24	300
25	1

TABLE III: COEFFICIENTS OF THE POWER SERIES EXPANSION OF G_x

m	p_m	$\frac{p_m}{m!}$	$\sum_{k=1}^{m} \frac{p_k}{k!}$
0	1.7182818285	---	---
1	0.6037828628	0.6037828628	.6037828628
2	.5483782849	.2741891425	.8779720053
3	.5429635131	.09049391887	.9684659241
4	.5857049319	.02440437216	.9928702963
5	.6823668995	.005686390829	.9985566871
6	.8481549252	.001177992952	.9997346801
7	1.111720107	.000220579385	.9999552594
8	1.522590134	.00003776265211	.9999930221
9	2.164362125	.000005964401799	.9999989865
10	3.177900086	.000000875744071	.9999998622
11	4.802247807	.000000120306433	.9999999826
12	7.447361538	.000000015547675	.9999999981
13	11.82463354	.000000001898923	1.0000000000
14	19.18356156	.000000000220050	1.0000000002
15	31.74519554	.000000000024276	1.0000000002

TABLE lV: G_{-n} UP TO G=20

n	G_{-n}
0	.6 321 205 588
1	.4 848 291 072
2	.4 217 734 383
3	.3 934 093 945
4	.3 801 978 350
5	.3 738 958 961
6	.3 708 415 557
7	.3 693 454 823
8	.3 686 075 427
9	.3 682 418 738
10	.3 680 601 230
11	.3 679 696 382
12	.3 679 019 333
13	.3 678 906 810
14	.3 678 850 591
15	.3 678 822 496
16	.3 678 808 447
17	.3 678 801 431
18	.3 678 797 922
19	.3 678 796 167
20	.3 678 794 412

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